ISSN: 2414-1895 DOI: 10.6919/ICJE.202209_8(9).0015

# Comparative Study of Riemann and Lebesgue Integrals

Guangbin Liu

School of Economics, Guangzhou College of Commerce, Guangzhou 511363, China

## Abstract

**The Riemann integral can be seen as making a division in the domain of definition of the integral, while the Lebesgue integral is making a division in the domain of values. When performing the division, the Riemann integral may result in huge amplitudes and, as a consequence, there is not Riemann integrability. This shortcoming is circumvented by the Lebesgue integral, which divides the value domain in such a way that the amplitude is satisfactory. In this paper, we introduce the concept of interval measure of the domain of definition of a function, and use the countable and additive properties of the Lebesgue integral to solve the Riemann integral problem for partially discontinuous functions. If the Riemann-producible functions are not closed to the limit operation, then it is possible to calculate the columns of those functions by exchanging the symbols for the integral and the limit. However, the conditions that must be met in order for integral and limit operations to be exchangeable are fairly stringent. The Lebesgue integral improves the solution to this problem by easing some of the conditions that must be met for exchangeability.**

## Keywords

**Riemann Integral; Lebesgue Integral; Definition Domain Partition; Value Domain Partition.**

## 1. Introduction

Wei, Y. et al. (2012), proved that: the Lebesgue integral of any non-negative Lebesgue integrabel function can be expressed as a Riemann integral of a monotonically decreasing function (including Riemann flaw integral, Riemann infinite interval integral); the integral of any Lebesgue integrable function can be expressed as the Riemann integral of two monotonically decreasing Riemann integral of the difference of two monotonically decreasing functions on (0,+∞), or a Riemann integral of a monotonically decreasing function on (-∞,0) and (0,+∞) [1].

Rongli Huang (2015) pointed out that Le Berger's improvement of the old theory of integration started from changing the steps required for the procedure of calculating definite integrals, and it was from this idea that he succeeded in promoting the Riemann integral. By comparing the advantages and disadvantages of Riemann integral to explore the central idea of Lebesgue's integral, students can deeply understand the framework and meaning of Lebesgue's integral [2].

Zhang Liying (2015) illustrated that the L-integral is not a generalization of the R-invariant integral by two examples, and also explored the relationship between the L-integral and the R-invariant integral, giving a sufficient condition between the L-integral and the R-invariant integral when the function satisfies certain conditions, which gives us a deeper understanding of the two integrals [3].

Liu Song (2016) elaborated the limitations of Riemann integral and the superiority of Lebesgue integral in terms of the continuity of the integrabel function,the integral limit theorem,the completeness of the space of integrabel functions and the fundamental theorem of calculus [4].

ISSN: 2414-1895 DOI: 10.6919/ICJE.202209_8(9).0015

Zhang, Y.L. and other researchers (2020), pointed out the relationship between the Lebesgue integral and the Riemann integral by comparing them analytically, and gave the condition that the Lebesgue integrabel function is Riemann integrabel [5].

# 2. Analysis and Argumentation

## 2.1 Definition of Riemann Integral

### 2.1.1 The Following First Defines a base $\mathfrak{B}$ on the Set X

Let X be a non-empty set and $\mathcal{B}\subset P(X)$ ,$P(X)$ be the power set of $X$ satisfying the following two conditions.

1) $\forall B\in\mathcal{B}$ which $B\neq\varnothing$ .
2) $\forall B_1\in\mathcal{B}$ ,$\forall B_2\in\mathcal{B}$ , (there exists $B\in\mathcal{B}$ , such that $B\subset B_1\cap B_2$ ).

At this point, $\mathcal{B}$ is said to be a base of the set $X$.

### 2.1.2 On the Limit of the Base

Let $X$ be a nonempty set, $\mathcal{B}$ be a base of $X$ ,$f:X\to\mathbb{R}$ , where $A\in\hat{\mathbb{R}}$ , an open neighborhood U of any A, and there exists $B\in\mathcal{B}$ such that $f(B)\in U$ , is said to be $f(x)$ about $B$ in the limit of $A$, denoted $\lim\limits_{\mathcal{B}} f(x)=A$, and interpreting the above equation in the definition of $\varepsilon-\delta$ ,$\forall\varepsilon>0$ ,there exists $B\in\mathcal{B}$ such that $|f(x)-A|<\varepsilon$ $\forall x\in B$.

### 2.1.3 Subdivision

Let $x_0,x_1,x_2,......,x_n\in[a,b]$ , satisfy: $a=x_0<x_1<...<x_n=b$ ,

Then call $\mathcal{G}=[x_{i-1},x_i]=\mathcal{I}_i$ (where $i=1,2,...,n$ ) as a subdivision of $[a,b]$, and the maximum interval length $\|\mathcal{G}\|$ of the subdivision G is called the granularity of the subdivision G and for each $i\in\{1,2,...,n\}$ , take $\xi_i\in\mathcal{I}_i$ , which the subdivision $(\mathcal{G},\xi)$ with sign points for the interval [a,b] is given, referred to as the division with marker $(\mathcal{G},\xi)$ , where $\xi=(\xi_1,\xi_2,...,\xi_n)$ , the following set is defined.

$\Omega=\{(\mathcal{G},\xi):(\mathcal{G},\xi)\text{ is a differentiation with maker points for }[a,b]\}$ $(\mathcal{G},\xi)$ differentiation with maker points $[a,b]$, $\forall\delta>0$, order $B_\delta=\{(\mathcal{G},\xi)\in\Omega:\|\mathcal{G}\|<\delta\}$.

The order $\mathcal{B}_\Omega=\{B_\delta\in P(\Omega):\delta>0\}$ , the following can be verified $\mathcal{B}_\Omega$ is a base of $\Omega$.

Verify that (i), $\forall\delta>0$ , we can take the n-equivalent partition of the interval $[a,b]$ , when n is sufficiently large such that $\|\mathcal{G}\|=\dfrac{b-a}{n}<\delta$ , so that the n-equivalent partition falls into $B_\delta$ , i.e. $B_\delta$ $\neq\varnothing$.

Verify that (ii), $\forall B_{\delta_1},B_{\delta_2}\in\mathcal{B}_\Omega$ , taken from $\delta=\min\{\delta_1,\delta_2\}$ , then $B_\delta=B_{\delta_1}\cap B_{\delta_2}\subset\mathcal{B}_\Omega$ , so $\mathcal{B}_\Omega$ is a base of $\Omega$.

### 2.1.4 Riemann Sum

If f(x) is defined on[a, b], ($\mathfrak{S}$ ,$\xi$ ) is a labeled subdivision and the Riemann sum is:

$$\sigma(f;\mathcal{G},\xi)=\sum_{i=1}^{n}f(\xi_i)|\mathcal{I}_i|$$

ISSN: 2414-1895 DOI: 10.6919/ICJE.202209_8(9).0015

### 2.1.5 Riemann Integral

If Riemann sum $\sigma(f;\mathfrak{S},\xi)$ along the basis $\mathfrak{B}_\Omega$ exists (finitely) in the limit of f(x), it is said that f(x) is$[a,b]$ f(x) is Riemann integrable over [a,b].

$$\int_a^b f(x)dx = \lim_{\mathcal{B}_\Omega} \sigma(f;\mathcal{G},\xi)$$

Since$\|\mathfrak{S}\| \to 0$ denotes$\mathfrak{B}_\Omega$ is natural, so the Riemann integral has another expression:

$$\int_a^b f(x)dx = \lim_{\|\mathcal{G}\|\to 0} \sum_{i=1}^{n} f(\xi_i)|\mathcal{I}_i|$$

This expression is the way most domestic mathematical analysis textbooks give the definition.

Theorem 1 The above two ways of Riemann integration are equivalent.

Proof: There exists $A \in \mathbb{R}$ such that $\lim_{\mathcal{B}_\Omega} \sigma(f;\mathcal{G},\xi) = A$.

$\Leftrightarrow$ The existence of , $A \in \mathbb{R}$ $\forall \varepsilon > 0$ , exists $B_\delta \in \mathcal{B}_\Omega$ , $\forall(\mathcal{G},\xi) \in B_\delta$ , $|\sigma(f;G,\xi) - A| < \varepsilon$ .

$\Leftrightarrow$The existence of , $A \in \mathbb{R}$ $\forall \varepsilon > 0$ , the existence of$\forall \delta > 0$, $\forall[a,b]$with the subdivisions$(\mathcal{G},\xi)$ , $\|\mathcal{G}\| < \delta$ , such that $|\sigma(f;G,\xi) - A| < \varepsilon$ .

$\Leftrightarrow$The existence of , $A \in \mathbb{R}$ $\forall \varepsilon > 0$ , the existence of $\forall \delta > 0$ , $\forall[a,b]$ subdivision $\mathcal{G}, \forall \xi_i \in \mathcal{I}_i$ $\|\mathcal{G}\| < \delta$ such that $|\sigma(f;G,\xi) - A| < \varepsilon$ .

$$\Leftrightarrow \lim_{\|\mathcal{G}\|\to 0} \sum_{i=1}^{n} f(\xi_i)|\mathcal{I}_i| = A$$

### 2.1.6 Darboux Grand Sum and Darboux Minor Sum

Assume that f(x) is a bounded function of [a,b] and subdivide [a,b] into $\mathcal{G}: a = x_0 < x_1 < \ldots < x_n = b$ , $M_i = \sup\{f(x): x \in [x_{i-1}, x_i]\}$ ,$m_i = \inf\{f(x): x \in [x_{i-1}, x_i]\}$ ,denoted$[x_{i-1}, x_i] = \Delta x_i$ , and$\lambda = \max[x_{i-1}, x_i]$ , where$1 \le i \le n$.

$$\overline{S}(P) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$\underline{S}(P) = \sum_{i=1}^{n} m_i \Delta_i$$

Darboux grand sum and Darboux minor sum, respectively.

### 2.1.7 Sufficient and Necessary Conditions for Riemann Integrability

The limit of Darboux major sum is equal to the limit of Darboux minor sum, i.e.

$$\lim_{\lambda\to 0} \overline{S}(P) = \lim_{\lambda\to 0} \underline{S}(P)$$

Volume 8 Issue 9, 2022
DOI: 10.6919/ICJE.202209_8(9).0015

## 2.2 Definition of the Lebesgue integral

Let $E$ be a measurable set and $m(E)<\infty$, and $f(x)$ be a measurable function on $E$ and is bounded. We can set $A \le f(x) \le B$ , take $C: A = y_0 < y_1 < \dots < y_n = B$ as any division of $[A,B]$ , write $\Delta y_i = [y_{i-1}, y_i]$ , and let $\delta(C) = \max \Delta y_i$ where $1 \le i \le n$ , take any $\xi_i \in E_i$ , and write the summation as:

$$S(C) = \sum_{i=1}^{n} f(\xi_i) m(E_i)$$

When $\lim_{\delta(C)\to 0} S(C)$ is a finite value, then $f(x)$ is said to be Lebesgue integrable on $E$

Of course, there are other ways to define the Lebesgue integral, for example, by first defining the Lebesgueer integral of a nonnegative simple measurable function, then deriving the Lebesgue integral of a nonnegative measurable function based on (the property that a nonnegative measurable function can be approximated by a column of nonnegative simple measurable functions), and finally using $f(x) = f^+(x) - f^-(x)$ , where $f^+(x)$ is the positive part of $f(x)$ , $f^-(x)$ is the negative part of $f(x)$ , and $f(x)$ is the measurable function on the measurable set $E$ to derive the general Lebesgue integral of the measurable function.

$\int_E f(x)dx = \int_E f^+(x)dx - \int_E f^-(x)dx$. Note that $\int_E f^+(x)dx$ and $\int_E f^-(x)dx$ cannot be positive or negative infinity at the same time, and if $\int_E f(x)dx$ is a finite real number, $f(x)$ is said to be Lebesgue integrable on $E$.

## 2.3 Comparison of Riemann Integral and Lebesgue Integral

### 2.3.1 Expansion of the Class of Integrabel Functions

Example 1 Dirichlet function, $D(x) = \begin{cases} 1, x \in Q \cap [0,1] \\ 0, x \in Q \setminus [0,1] \end{cases}$.

It is known that D(x) has amplitude 1 on each small interval of $[0,1]$ and the amplitude on each small interval is too large to be Riemann-integrable, according to the additivity of the region of the Lebesgue integral, $\int_{[0,1]} D(x)dx = \int_{[0,1]\cap Q} D(x)dx + \int_{[0,1]\setminus Q} D(x)dx = 0$ , so D(x) is Lebesgue integrable.

Lemma 1 $f(x)$ is continuous on the measurable set $E \Longrightarrow f(x)$ is a measurable function.

Proof: According to the equivalent inscription of function continuity, the open set in $O$ , $\forall \mathbb{R}$ makes $f^{-1}(O)$ an open set. Since the open set in $\mathbb{R}$ can be written as a sum of countable two non-intersecting open intervals, each open interval is a measurable set, then the open set in $\mathbb{R}$ is also a measurable set, so $f^{-1}(O)$ is a measurable set, so $f(x)$ is a measurable function.

Theorem 2 ( Lebesgue's criterion for Lebesgue integrable) $f(x)$ is a bounded function on $[a,b]$ and $f(x)$ is Riemann-integrable on $[a,b]$ $\Leftrightarrow f(x)$ is the discontinuity at the set of zero measurements of $[a,b]$.

Theorem 2 can be used to quickly determine the integrability of the Riemann function, because the Riemann function is continuous at irrational points and discontinuous at rational points, but the measure of the rational points is zero, so the Riemann function is Riemann-integrable.

Theorem 3( Lebesgue integral is a generalization of the Riemann integral) If $f(x)$ is Riemann-integrable on $[a,b]$ , then $f(x)$ is Lebesgue-integral on $[a,b]$ Proof: Since $f(x)$ is Lebesgue integrable on $[a,b]$ , by Theorem 2, the discontinuity of on $f(x)$ $[a,b]$ is the zero measure set, and let the zero measure subset of $[a,b]$ be $Z$ , then $f(x)$ is continuous on $[a,b]\setminus Z$ , and by Lemma 1, $f(x)$ is a continuous function continuous on $[a,b]\setminus Z$ , and since $f(x)$ is Riemann-integrable on $[a,b]$ , then $f(x)$ is Riemann-integrable on $[a,b]\setminus Z$ , then $f(x)$ is bounded on $[a,b]\setminus Z$ , and by the monotonicity of the Lebesgue integral, $f(x)$ is Lebesgue-integrable on $[a,b]\setminus Z$ , and since the zero measure subset does not change the value of the Lebesgue integral, $f(x)$ is Lebesgue integrable on $[a,b]$.

However, in general, the Lebesgue integral is not a generalization of the Riemann integral.

Example 2 Assume that $f(x)=\dfrac{\sin x}{x}$ , $x>0$ , $f(x)=1$ , $x=0$ .

(R) $\int_0^{+\infty} f(x)dx=\dfrac{\pi}{2}$ , (L) $\int_{[0,+\infty)} f(x)dx=\infty$ .

The function is Riemann integrable, but the fuction is not Lebesgue integrable.

Theorem 4 (Ruzin's Theorem) If $f(x)$ is a finite measurable function on $E\subset\mathbb{R}^n$ , almost everywhere, $\forall\delta>0$ , there exists a closed set $F$ , $m(E\setminus F)<\delta$ in $E$ such that $f(x)$ is a continuous function on $F$

By Theorem 3, we can indeed Riemann-producible functions must be Lebegg-producible, thus expanding the class of productible functions. From Theorem 4, almost everywhere finite measurable functions are essentially continuous functions, then Lebegg-producible functions are essentially continuous functions, expanding the class of productible functions.

### 2.3.2 Exchange of Integrals and Limits

The Riemann integral can be said to have a fatal flaw in that it is not closed to limit operations and requires the addition of some stronger conditions to ensure that the limits and integrals are exchangeable in order, whereas the Lebesgue integral is obtained by controlling the convergence theorem, where the limits and integrals are exchangeable, and the conditions in the control convergence theorem are relatively weak.

Theorem 5 Let each term of the function column $\{f_n(x)\}$ $f_n(x)$ be continuous on $[a,b]$ and converge uniformly on $[a,b]$ , then:

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_b^a \lim_{n\to\infty} f(x)dx$$

Theorem 6 ( Dominated convergence theorem) Let $E$ be a measurable set and let $f(x)$ and $f_n(x)$ both be measurable functions on $E$ , if the following conditions hold:

$\{f_n(x)\}_{n=1}^{\infty}$ On $E$ a.e converges to $f(x)$ .

There exists $g\in L(E)$ such that $|f_n(x)|\le g(x)$ , a.e, $n=1,2,\ldots$

Then both $f(x)$ and $f_n(x)$ are integrabel, and we have:

$$\lim_{n\to\infty}\int_E |f_n(x)-f(x)|dx=0$$

thereby having:

ISSN: 2414-1895
DOI: 10.6919/ICJE.202209_8(9).0015

$$\lim_{n\to\infty}\int_E f_n(x)dx = \int_E f(x)dx = \int_E \lim_{n\to\infty} f_n(x)dx$$

### 2.3.3 Absolute Integrability

Theorem 7 $f(x)$ is a measurable function on $E$, $f \in L(E) \Leftrightarrow |f| \in L(E)$.

Note: Since $\int_E f^+(x)dx$ and $\int_E f^-(x)dx$ cannot be positive or negative infinity at the same time, at least one of them is a finite real number.

Proof: $f \in L(E) \Leftrightarrow \int_E f(x)dx = \int_E f^+(x)dx - \int_E f^-(x)dx \in \mathbb{R} \Leftrightarrow \int_E f^+(x)dx \in \mathbb{R}, \int_E f^-(x)dx \in \mathbb{R} \Leftrightarrow$ $f^+(x) \in L(E)$, $f^-(x) \in L(E)$.

$\Leftrightarrow \int_E |f(x)|dx = \int_E f^+(x)dx - \int_E f^-(x)dx \in \mathbb{R} \Leftrightarrow \int_E |f(x)|dx \in \mathbb{R}$.

This result does not necessarily hold for Riemann integration, for example, let the functions $F(x) = \begin{cases} 1, & x \in Q \cap [0,1] \\ -1, & x \in Q \setminus [0,1] \end{cases}$, be obvious that $F(x)$ is not Riemann integrable, but $|F(x)|$ is Riemann integrable.

### 2.3.4 Countable Additivity of the Integration Region

Theorem 8, $E = \bigcup_{n=1}^{\infty} E_n$, $E_n$ are two non-intersecting measurable sets, if $f(x)$ on $E$ is nonnegative measurable or integrable, then:

$$\int_E f(x)dx = \int_{\bigcup_{n=1}^{\infty} E_n} f(x)dx = \sum_{n=1}^{\infty}\int_{E_n} f(x)dx$$

Proof: Let $f(x)$ on $E$ is non − negative measurable function, such that $f_n(x) = \begin{cases} f(x), & x \in \bigcup_{n=1}^{p} E_n \\ 0, & x \in E \setminus \bigcup_{n=1}^{p} E_n \end{cases}$.

Then $\{f_n(x)\}_{n=1}^{\infty}$ is a non-negative measurable function sequence on $E$, and it converges with f on $E$ incrementally. According to Levi's monotone limit convergence theorem,

$$\int_E f(x)dx = \lim_{n\to\infty}\int_E f_n(x)dx = \lim_{n\to\infty}\int_{\bigcup_{n=1}^{p} E_n} f_n(x)dx = \lim_{n\to\infty}\int_{\bigcup_{n=1}^{p} E_n} f(x)dx = \lim_{n\to\infty}\sum_{n=1}^{p}\int_{E_n} f(x)dx = \sum_{n=1}^{\infty}\int_{E_n} f(x)dx$$

where the first equal sign uses Levi's monotone limit convergence theorem, and the fourth equal sign uses finite additivity, and then look at the case of setting $f(x)$ is on $E's$ the integrable case.

$$\int_E f(x)dx = \int_E f^+(x)dx - \int_E f^-(x)dx = \sum_{n=1}^{\infty}\int_{E_n} f^+(x)dx - \sum_{n=1}^{\infty}\int_{E_n} f^-(x)dx = \sum_{n=1}^{\infty}\int_{E_n}[f^+(x) - f^-(x)]dx = \sum_{n=1}^{\infty}\int_{E_n} f(x)dx$$

Then the Lebesgue integrable consists of countable additivity, while the Riemann integral does not have countable additivity, but the Riemann integral has finite additivity.

Example 3 Set $f(x)=1$ $E=(0,1]$ $E_i=(\frac{1}{1+i},\frac{1}{i}]$ $i=1,2,\cdots$ $E^*=(\frac{1}{n+1},1]$

$E=\bigcup_{i=1}^{\infty}E_i$ , $E^*=\bigcup_{i=1}^{n}E_i$ $E_i\cap E_j=\varnothing$, $i\neq j$.

$\int_{E^*}f(x)dx=\frac{n}{n+1}$ , $\sum_{i=1}^{n}\int_{E_i}f(x)dx=1$ (finite additivity example).

$\sum_{i=1}^{\infty}\int_{E_i}f(x)dx=\int_{E_1}f(x)dx+\int_{E_2}f(x)dx+\cdots=1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}\neq 1$ (No countable additivity).

The Riemann integral is based on the Jordan measure, which has only finite additivity, while the Lebesgue integral is based on the Lebesgue measure, which has countable additivity, which is reflected in the fact that the Riemann integral has only finite additivity and the Lebesgue integral has countable additivity.

## References

[1] Y. Wei, B.L. Zhang: The relationship between Lebesgue and Riemann integrals from a new perspective Journal of Southwest China Normal University (Natural Science Edition) 2012,37(10) p. 6-9.

[2] Y.L. Zhang, F. Huang: the relationship between the Lebesgue integral and the Riemann integral Journal of Jiaozuo Teachers College 2020,36(01) p. 70-73+76.

[3] S. Liu: Limitations of Riemann integral and superiority of Lebesgue integral Journal of Hefei University (Comprehensive ED) 2016,33(04) p. 14-16+34.

[4] R. L. Huang: Exploring the teaching of Le Berger's integral ideas in university mathematics majors Survey of Education 2015,4(02) p. 43-44.

[5] L. Y. Zhang: An analysis of the correlation between the Lebesgue integral and the Riemann anomaly integral Journal of Guangdong Polytechnic Normal University 2015,36(11) p. 43-44.