ISSN: 2414-1895 DOI: 10.6919/ICJE.202109_7(9).0047

Research and Extension of Logistic Equation

Maozhi Tang

College of Engineering and Physical Sciences, University of Birmingham, Birmingham, B15 2TT, UK.

Abstract

This paper shows: First, show the behaviour of the equation f(x)=ax(1-x) when 'a' belongs to dif-ferent sub-intervals in [0,4], and see the sequence $U_{n+1}=aU_n(1-U_n)$ Second, give the proof of Sharkovskii's theorem, which is if a continuous function has a 3-cycles in a given interval, then there must be an n-cycles in this interval for any natural number n>0. Third, focus on n-cycles $f^n(x_0)=x_0$ with cycles x_0, x_1,..., x_{n-1} and prove that D $f^n(x_k)=f'(x_0)f'(x_1)...f'(x_{n-1})$ for any k in {1,2,...,n-1}. Finally, this paper shows the proof of the Coppel theorem for the sequence given by $U_{n+1}=f(u_n)$, that is, if the function f is continuous and bounded and has no 2-cycles, then the sequence must converge to a fixed point in the do-main.

Keywords

Convergence; Logistic Equation; Sharkovskii's Theorem; Coppel Theorem.

1. Introduction

Logistic equation was first introduced by mathematical biologist Pierre -Francois Verhulst in 1839. [1]

The equation f(x)=ax(1-x) can be used to form a sequence $U_{n+1}=aU_n(1-U_n)$, but in order to assure $U_n > 0$ regardless of the initial condition, we can assume a $\in$ [0,4]. Sharkovskii's theorem is an extension from the logistic equation with cycles. And Coppel theorem is the form of logistic equation but its right-hand side is some arbitrary continuous function 'f', which means the theorem focus on the equation $U_{n+1}=f(u_n)$.

2. Behavior of the equation f(x)=ax(1-x), a $\in$ [0,4]

The conclusion of the sequence will be given without the proof.

CONCLUSION

When a belongs to (0,1), the sequence converges to 0.

When a belongs to (1,3), it converges to L. [2]

A fixed point of 'f' is a point x_0 such that $f(x_0)=x_0$.

There are two fixed points, 0 and 1-1/a, we use L to represent 1-1/a.

When a is greater than 3, *the sequence converges to cycles when a is close to 3, and tends to chaos when a tends to 4.*

3. Sharkovskii's theorem

Before this theorem is proved, there are some lemmas that are useful. There is not proof of lemma 1 and 2 in this paper, but lemma 3 is proved.

3.1 Lemma 1:

Suppose a function f: I → I is continuous. Suppose J = [a,b] ⊂ I.

Volume 7 Issue 9, 2021
DOI: 10.6919/ICJE.202109_7(9).0047

If $f(J) \supset J$, then there exists a fixed point $x_0 \in J$

3.2 Lemma 2:

Suppose a function $f: I \to I$ is continuous. Suppose J_1, J_2 are two closed intervals on I. If $f(J_1) \supset J_2$, then there exists an interval $K \subset J_1$, such that $f(K) = J_2$

3.3 Lemma 3:

Suppose a function $f: I \to I$ is continuous. Suppose $J_0, J_1, \ldots, J_{n-1}$ are

n closed intervals on I. If $f(J_0) \supset J_1$, $f(J_1) \supset J_2, \ldots, f(J_{n-2}) \supset J_{n-1}$, $f(J_{n-1}) \supset J_0$, then there exists $x_0 \in I$, such that $f^n(x_0)=x_0$, and $f^i(x_0) \in J_i$ for each $i \in \{1,2,\ldots,n-1\}$ [3].

Proof:

Given that $f(J_{n-1}) \supset J_0$, using Lemma 2, there exists an interval $K_{n-1} \subset J_{n-1}$ $(\#_{n-1})$ such that $f(K_{n-1})=J_0$.

Because $f(J_{n-2}) \supset f(J_{n-1})$ and $(\#_{n-1})$, $f(J_{n-2}) \supset K_{n-1}$.

So there is an interval $K_{n-2} \subset J_{n-2}$ $(\#_{n-2})$ such that $f(K_{n-2}) = K_{n-1}$ by Lemma 2.

Repeat this process until $f(J_0) \supset f(J_1)$, and the interval $K_0 \subset J_0$ such that $f(K_0) = K_1$

So there are

$f(K_0) = K_1$ $\quad K_0 \subset J_0$ $\quad (*)$

$f(K_1) = K_2$ $\quad K_1 \subset J_1$

$f(K_2) = K_3$ $\quad K_2 \subset J_2$

…

$f(K_{n-2}) = K_{n-1}$ $\quad K_{n-2} \subset J_{n-2}$

$f(K_{n-1}) = J_0$ $\quad K_{n-1} \subset J_{n-1}$

Combine these equations,

$f^n(K_0) = J_0 \supset K_0$, by $(*)$, which is $f^n(K_0) \supset K_0$

Therefore, by Lemma 1, there is a fixed point $x_0 \in K_0$, which means

$f^n(x_0) = x_0$ and $f^i(x_0) \in f^i(K_0) \in J_i$

End of proof

Now prove the Sharkovskii's theorem: If a continuous function has a 3-cycles in a given interval, then there must be an n-cycles in this interval for any natural number n>0. [4]

Suppose the 3-cycles is a, b, c, which means $f(a) = b$, $f(b) = c$, $f(c) = a$. We adjust the order of these points by making them $a<b<c$, or $a>b>c$. The case $a<b<c$ will be proved, and the other case can be proved in a similar way.

Suppose $f: I \to I$ is continuous, and $[a,c] \subset I$.

Suppose $J = [a,b]$, $K = [b,c]$. So $f(J) \supset K$, and $f(K) \supset [a,c] = J \cup K$ (&)

First, if n=1, by (&), $f(K) \supset K$, based on Lemma 1, there exists a fixed point in K, which can be called a 1-cycle.

Second, if n=2, by (&), $f(K) \supset J$, $f(J) \supset K$. In Lemma 3, suppose $J_0 = K$, $J_1 = J$. So there exists $x_0 \in K_0 \subset J_0 = K = [b,c]$, such that $f^2(x_0) = x_0$, and $f(x_0) \in J_1 = J = [a,b]$.

Suppose $f(x_0) = t$. So x_0, t is a 2-cycles if an only if x_0 is not equal to t. Prove this by looking for a contradiction.

Assume $x_0=t$, it is already proved that $x_0 \in [b,c]$ and $t \in [a,b]$, by assumption there is only one case that $x_0=t=b$. But $f(b) = c$ because a,b,c is a 3-cycles, there is a contradiction with $x_0=t$, so x_0 is not equal to t.

Third, if n=3, use lemma 3.

ISSN: 2414-1895 DOI: 10.6919/ICJE.202109_7(9).0047

Define (n-1) intervals $I_0 = I_1 = \ldots = I_{n-2} = K = [b,c]$,and $I_{n-1} = J = [a,b]$

$f(I_j) = f(K) \supset K = I_{j+1}$ for $j = 0,1,\ldots,n-3$

$f(I_{n-2}) = f(K) \supset [a,c] = J \cup K \supset J = I_{n-1}$ by (&)

$f(I_{n-1}) = f(J) \supset K = I_0$

So by lemma 3, there exists $x_0 \in K_0$ (in lemma 3) $\subset I_0 =[b,c]$, such that

$f^n(x_0) = x_0$ and $f^j(x_0) \in I_j$ for any j in $\{0,1,2,\ldots,n-1\}$

Also we have to prove that n is the shortest period, i.e. there is no repetition in this n-cycle. Like the second part, prove this by contradiction.

Suppose n is not the shortest period, there must be some element

$u \in \{x_0, f(x_0),\ldots,f^{n-2}(x_0)\}$ such that $f^{n-1}(x_0) = u$ ($)

Since $f^j(x_0) \in I_j$ for any j in $\{0,1,2,\ldots,n-1\}$, $f^{n-1}(x_0) \in I_{n-1} = [a,b]$. Also $u \in I_m$ for some m in $\{0,1,\ldots,n-2$. These intervals are the same, which means

$u \in [b,c]$.

By ($), the only case is $f^{n-1}(x_0) = u = b$. In that way

$x_0 = f^n(x_0) = f(b) = c$, and $f(x_0) = f(c) = a$ does not belong to [b,c], which contradicts $f^j(x_0) \in I_j$ for any j in $\{0,1,2,\ldots,n-1\}$. So there is no repetition in this cycle.

End of proof

4. Derivative of $f^n(x_k)$

Conclusion: Suppose $x_0, x_1,\ldots, x_{n-1}$ is an n-cycles of the function f(x). The derivative D $f^n(x_k)=f'(x_0)f'(x_1)\ldots f'(x_{n-1})$ for any k in $\{1,2,\ldots,n-1\}$

Proof: Define $F(x) = f^n(x)$ There is

$F(x_0) = f^n(x_0) = x_0$

$F(x_1) = f(f^n(x_0)) = f(x_0) = x_1$

…

$F(x_{n-1}) = f^n(x_{n-1}) = x_{n-1}$

So $(x_0,F(x_0))$, $(x_1,F(x_1).)$,…, $(x_{n-1}, F(x_{n-1}))$ are all fixed points in F(x)

First, focus on $DF(x_0)$

$F'(x_0) = [\, f^n(x_0)]' = [\, f(f^{n-1}(x_0))\,]'$

Notice that $f^{n-1}(x_0) = x_{n-1}$ By the rule of derivative of composite functions,

$F'(x_0) = f'(x_{n-1})*(x_{n-1})' = f'(x_{n-1})*(f^{n-1}(x_0)) = f'(x_{n-1})*(f(f^{n-2}(x_0))' = f'(x_{n-1})*(f(x_{n-2}))' = f'(x_{n-1})* f'(x_{n-2})*(x_{n-2})' = \ldots = f'(x_0)f'(x_1)\ldots f'(x_{n-1})$

Second, look at $DF(x_1)$, it is in a similar way

$F'(x_1) = (f^n(x_1))' = [f(f^{n-1}(x_1))]'$

Notice that $f^{n-1}(x_1) = x_0$ and by rule of derivative of composite functions,

$F'(x_1) = f'(x_0)*(f^{n-1}(x_1))' = f'(x_0)*[f(f^{n-2}(x_1)]' = f'(x_0)*f'(x_{n-1})*\ldots*f'(x_1) = F'(x_0)$

Third for any k in $\{1,2,\ldots,n-1\}$, there is a similar way of calculating

D $f^n(x_k)$, and they are all the same.

Therefore, the derivative D $f^n(x_k)=f'(x_0)f'(x_1)\ldots f'(x_{n-1})$ for any k in $\{1,2,\ldots,n-1\}$

End of proof

5. Coppel theorem and its proof

Before the proof of the Coppel theorem, there is a preliminary theorem that is useful for the proof.

ISSN: 2414-1895 DOI: 10.6919/ICJE.202109_7(9).0047

5.1 Preliminary theorem:

Given a continuous function f: [a,b] $\rightarrow$ [a,b] and a sequence $U_{n+1}=f(u_n)$. If f has no 2-cycles, then for any c $\in$ [a,b] and a natural number n>0, $f^n(c)$ >,<,= c is according f(c) >,<,= c

Prove this theorem by induction:

First: When n = 1, this theorem holds obviously.

Second: Assume this theorem holds when n $\leq$ m, with m>0 a natural number. That is, if f has no 2-cycles, then for any c $\in$ [a,b], $f^m(c)$ >,<,= c is according f(c) >,<,= c

Third: Look for the case n = m+1

Firstly prove that if $f^{m+1}(c) = c$, then f(c) = c. Try to look for a contradiction if f(c) > or < c.

Suppose $f^{m+1}(c) = c$ (^)

a) Suppose f(c) > c (&). There is $f^m(c) > c$ by induction hypothesis. Next prove d < f(c) under this case.

Assume $d = f^m(c) = f^{m-1}(f(c)) \geq f(c)$ (!) and then look for a contradiction.

By (!), $f^{m-1}(f(c)) \geq f(c)$,

so $f(f(c)) \geq f(c)$ by induction hypothesis in the second step.

Also use this step, $f^m(f(c)) \geq f(c)$.

By (^), $f^{m+1}(c) = c$, so $c \geq f(c)$.

This contradicts to (&).

So d < f(c).

Next try to find a contradiction to disprove (&).

It is already known that d< f(c) , $d = f^m(c) > c$ by induction hypothesis, and $f(d) = f^{m+1}(c) = c$

So f(c) - d > 0, and f(d) - d < 0

Because f is a continuous function, there must be at least one point in [c,d] satisfying f(x) = d. Choose the closest point to c and name it e with e $\in$ [c,d]

There is f(e) = d and f(c) > d

and in the interval (c,e) , f(x) > d > x. (1)

Notice that f(f(c)) > c since f(c) > c and by induction hypothesis,

And $f(f(e)) = f(d) = f^{m+1}(c) = c < e$

Because f is continuous on (c,e), there must be at least one point satisfying f(f(x)) = x, but in this region f(x) > x by (1), so f(f(x)) can not be equal to x by induction hypothesis. It contradicts to (1). Therefore the whole assumption is inproper.

Thus, the case f(c) > c cannot be true.

Besides, the case f(c) < c is similar to f(c) > c and it cannot be true as well.

Therefore, f(c) = c.

Secondly prove $f^{m+1}(c)$ >,< c is according f(c) >,< c

The proof of: If $f^{m+1}(c) < c$, then f(c) < c. I assume f(c) > c is covered in this paper and the other cases can be proved in a similar way.

Suppose f(c) > c. Then $f^m(c) > c$ by induction hypothesis.

Because f: [a,b] $\rightarrow$ [a,b], $f^{m+1}(a) \geq a$. With $f^{m+1}(c) < c$, there must be some point satisfying $f^{m+1}(x) = x$. Choose the closest one to c and name it d. So in the interval (d,c] $f^{m+1}(x) < x$ (2)

It is proved that if $f^{m+1}(x) = x$, then f(x) = x. And by induction hypothesis, $f^m(x) = x$. So f(d) = d and f(x) > x, (3) in the interval (d,c] because there is no more point satisfying f(x) = x, f is continuous and f(c) >c.

So far, there are f(d) = d, f(c) >c, so there must be some point e in (d,c) such that f(e) $\in$ (d,c).

ISSN: 2414-1895 DOI: 10.6919/ICJE.202109_7(9).0047

Using (3), $f(e) > e$, $f(f(e))>f(e)$

So by induction hypothesis, $f^m(f(e)) > f(e) > e$, which means $f^{m+1}(e) > e$. It contradicts (2). So the assumption is not true.

Therefore, $f(c) < c$.

The case if $f^{m+1}(c) > c$, then $f(c) > c$ can be proved in a similar way.

End of proof

5.2 Coppel theorem:

Given a sequence $U_{n+1}=f(u_n)$, if the function $f: [a,b] \to [a,b]$ continuous and has no 2-cycles, then the sequence must converge to a fixed point in the domain. [5]

Proof:

a) If u_n is monotone, either increasing and decreasing, by the monotone convergence principle, it converges to some point L, and the right and left side of the equation $U_{n+1}=f(u_n)$ should be the same. So $L = f(L)$, which means L is a fixed point.

b) If u_n is not monotone, there is infinite amount of n,m, such that $U_{n+1} > U_n$ and $U_{m+1} < U_m$.

Establish two new sequences

$a_n = U_{n1}, U_{n2}, U_{n3},\dots$

$b_n = U_{m1}, U_{m2}, U_{m3},\dots$

Then it can be proved that a_n is monotone increasing:

Suppose two neighboring terms of a_n is $f^k(U_{n1})$, $f^p(U_{n1})$

There is $f(f^k(U_{n1})) > f^k(U_{n1})$ (4) , $f^p(U_{n1}) = f^{p-k}(f^k(U_{n1}))$

Combine (4) with preliminary theorem, there is $f^{p-k}(f^k(U_{n1})) > f^k(U_{n1})$, which means $f^p(U_{n1}) > f^k(U_{n1})$. Therefore, a_n is monotone increasing

For the same reason, b_n is monotone decreasing. Assume a_n takes p as its limit, b_n takes q as its limit. There will be some point that $p = f(q)$, and $q = f(p)$ since there are no 2-cycles, $p = f^2(p)$ and by preliminary theorem, $f(p) = p$ is a fixed point.

6. Conclusion

The behavior of logistic equation is given in this paper, together with three lemmas to prove Sharkovskii's theorem, and the Coppel's theorem's proof using the preliminary theorem. All of them are the simplest in their theorem and there are more about them in other papers.

References

[1] EW Weisstein, 2003, Logistic Equation, https://mathworld.wolfram.com/LogisticEquation.html

[2] Maozhi Tang, part of the convergence of logistic equation, university of Birmingham student work, pp 01-05.

[3] Zhang Jingzhong, Xiong Jincheng, 1992.02, Sharkovskii theorem, https://wenku.baidu.com/view/b247dd547ed5360cba1aa8114431b90d6d85895d.

[4] Lin Zhixiong, The unicity of Sharkovskii-series(s-series), Journal of Fujian University of Technology, 2005, (03), P1.

[5] Keith Burns, Boris Hasselblatt, the Sharkovsky theorem: a natural direct proof, https://www.math.arizona.edu/~ dwang/BurnsHasselblattRevised-1.pdf.