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Spectra Nonrealizable by 5×5 Nonnegative Symmetric Matrices

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Abstract

The Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP) concerns about the necessary and sufficient conditions for a given spectrum to be realizable by a nonnegative symmetric matrix. This paper concerns about the case n=5 where a small region still remains unresolved. In this paper, certain 5-spectra are shown to be nonrealizable by extending a known method to a type more generalized spectra. This paper also gives an insight on how to use this method in obtaining better results with more precision in calculation or apply it to another similar case.

Keywords

Symmetric Matrix, Realizability, Inverse Eigenvalue Problem.

1. Introduction

The *Nonnegative Inverse Eigenvalue Problem* (*NIEP*) is about determining the necessary and sufficient conditions for a list $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ to be the spectrum of a nonnegative matrix. If there exists such matrix, namely A, then the list Λ is said to be *realizable* and the matrix A is a *realizing* matrix. In particular, the *SNIEP* considers a special case in which A is symmetric and call such list Λ *symmetrically realizable*. For $n \leq 4$, the *SNIEP* is exactly the same as the *real nonnegative inverse eigenvalue problem* (*RNIEP*), a special case of *NIEP* when the realizing matrix is nonnegative and real, and has been resolved (see [1,2]). A huge chunk of the *SNIEP* for $n = 5$ has been discussed in [3-6].

Our aim in this paper is to extend the idea in [3] to exclude more 5-spectra with three positive and two negative elements that are not symmetrically realizable. The main progress is done by relaxing the constraints on one of the variables in the spectrum.

2. Preliminaries

Theorem 1 (*Perron-Frobenius Theorem*)

Let A *be a nonnegative matrix and* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *be its spectrum. Then* $\rho(\mathrm{A}) = \max_i |\lambda_i| \in \Lambda$. *Here* $\rho(\mathrm{A})$ *is called the Perron root of* A.

See [7] for the case where A is positive and [8] for the extension to the nonnegative case.

Theorem 2

Let $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *and* λ_1 *be the spectral radius. Then for* $n \leq 4$, Λ *is symmetrically realizable if and only if* $\sum_{i=1}^{n} \lambda_i \geq 0$.

See Theorems 2.1 through 5.1 in [9] for constructions of realizing matrices for corresponding cases.

Theorem 3

If a list $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *is realizable, then it satisfies the following inequality (JLL Inequality)*

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$$\left(\sum_{i=1}^{n}\lambda_i^k\right)^m \le n^{m-1}\sum_{i=1}^{n}\lambda_i^{km}, k,m=1,2,\cdots \tag{1}$$

See [1,10] for the proof.

Lemma 4

If A *is a symmetric matrix with spectrum* $\Lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$, *then*

$$\mathrm{Tr}(\mathrm{A})=\sum_{i=1}^{n}\lambda_i\,. \tag{2}$$

Pf: Since A is symmetric, there is an invertible matrix P satisfying $\mathrm{A}=\mathrm{PDP}^{-1}$ where $\mathrm{D}=\mathrm{diag}\,\{\lambda_1,\cdots,\lambda_n\}$. Then $\mathrm{Tr}(\mathrm{A})=\mathrm{Tr}(\mathrm{PDP}^{-1})=\mathrm{Tr}(\mathrm{P}^{-1}\mathrm{PD})=\mathrm{Tr}(\mathrm{D})=\sum_{i=1}^{n}\lambda_i$.

Theorem 5 (*Cauchy Interlacing Theorem*)

If A *is a symmetric real matrix, and* B *is a principal submatrix of* A, *then the eigenvalues of* B *interlace the eigenvalues of* A.

A short proof of a generalized version of this theorem with A being Hermitian is given in [11].

Theorem 6

Let $0<\alpha,\beta$ *such that* $\alpha+\beta,2\beta<1<\alpha+2\beta$. *If* $2(\alpha+\beta)^3\ge 1+\alpha^3+(\alpha+2\beta-1)^3$, *then* $\{1,\alpha,\alpha,-(\alpha+\beta),-(\alpha+\beta)\}$ *is not realizable by a 5×5 symmetric nonnegative matrix.*

A proof of this theorem using Lemma 4 and Theorem 5 is provided in [3].

3. Main Result

Now the aim is to extend the idea used in proving Theorem 6 to the following list of

$$1,\alpha,\alpha-\varepsilon,-(\alpha+\beta),-(\alpha+\beta),$$

satisfying

$$0<\alpha,0<\beta,0<\varepsilon<\frac{4+4\sqrt{30}}{29}\alpha, \tag{3}$$
$$0<\varepsilon<1-2\beta<\alpha<1-\beta.$$

Assume that this list can be realized by a nonnegative symmetric matrix $\mathrm{A}=(a_{ij})$. Let A_i denote the 4-by-4 principal submatrix of A obtained by deleting the i-th row and column from A. By Theorem 5, the spectrum of A_i, $\sigma(\mathrm{A}_i)=\{p_i,\alpha-\delta_i,q_i,-(\alpha+\beta)\}$, satisfies

$$1\ge p_i\ge\alpha\ge\alpha-\delta_i\ge\alpha-\varepsilon\ge q_i\ge-(\alpha+\beta). \tag{4}$$

From Theorem 1, the following also holds

$$p_i\ge\alpha+\beta. \tag{5}$$

Since A is nonnegative, A_i must also be nonnegative and $\mathrm{Tr}(\mathrm{A}_i)\ge 0$. By (2),

$$p_i+q_i-\beta-\delta_i\ge 0. \tag{6}$$

Since $a_{ii}=\mathrm{Tr}(\mathrm{A})-\mathrm{Tr}(\mathrm{A}_i)\ge 0$,

$$1-p_i-q_i-\beta-\varepsilon+\delta_i\ge 0. \tag{7}$$

From properties $4\mathrm{Tr}(\mathrm{A})=\sum_{i=1}^{5}\mathrm{Tr}(\mathrm{A}_i)$ and $4\mathrm{Tr}(\mathrm{A}^3)\ge\sum_{i=1}^{5}\mathrm{Tr}(\mathrm{A}_i^3)$ which comes from (1) and (2), it follows that

$$4-3\beta-4\varepsilon=\sum_{i=1}^{5}(p_i+q_i-\delta)\,, \tag{8}$$

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$$4+4\alpha^3-3(\alpha+\beta)^3+4(\alpha-\varepsilon)^3 \geq \sum_{i=1}^{5}[p_i^3+q_i^3+(\alpha-\delta_i)^3]. \tag{9}$$

Now $\sum_{i=1}^{5}[p_i^3+q_i^3+(\alpha-\delta_i)^3]$ can be minimized to determine the conditions under which the inequality (9) will be violated with previous constraints. Fix $\sum_{i=1}^{5}p_i$, then $\sum_{i=1}^{5}p_i^3$ is minimized when $p_1=\cdots=p_5$. According to (5), it is legitimate to take $p=\alpha+\beta+t$ where $0\leq t\leq 1-\alpha-\beta$. Then from (8), it follows that

$$\sum_{i=1}^{5}q_i=4-3\beta-4\varepsilon+\sum_{i=1}^{5}(\delta_i-p_i)\leq 4(1-2\beta)+\varepsilon-5(\alpha+t)<0. \tag{10}$$

From (6) and (7), q_i can be bounded be the following

$$-(\alpha+t)+\varepsilon\leq q_i\leq 1-2\beta-(\alpha+t). \tag{11}$$

Since (10), at least one of q_i is needed to be as small as possible to minimize $\sum_{i=1}^{5}q_i^3$. Without loss of generality, take $q_1=-(\alpha+t)+\varepsilon$ and

$$q_2=\cdots=q_5=1-2\beta-(\alpha+t)-\frac{1}{4}\sum_{i=1}^{5}(\varepsilon-\delta_i)\leq 1-2\beta-\alpha-t.$$

Let $x=\sum_{i=1}^{5}\frac{\varepsilon-\delta_i}{4}$ and $f(t)=\sum_{i=1}^{5}[p_i^3+q_i^3+(\alpha-\delta_i)^3]$, then

$$f(t)=5(t+\alpha+\beta)^3+(\varepsilon-t-\alpha)^3+4\left(1-t-2\beta-\alpha-\frac{x}{4}\right)^3+\sum_{i=1}^{5}(\alpha-\delta_i)^3, \tag{12}$$

$$f'(t)=15(t+\alpha+\beta)^2-3(\varepsilon-t-\alpha)^2-12\left(1-t-2\beta-\alpha-\frac{x}{4}\right)^2. \tag{13}$$

From (3), it can be shown that f' is positive and increasing when $0\leq t\leq 1-\alpha-\beta$. Thus, $\min\{f(t): 0\leq t\leq 1-\alpha-\beta\}$ is obtained at $t=0$ and

$$\sum_{i=1}^{5}[p_i^3+q_i^3+(\alpha-\delta_i)^3]\geq 5(\alpha+\beta)^3+(\varepsilon-\alpha)^3+4\left(1-2\beta-\alpha-\sum_{i=1}^{5}\frac{\varepsilon-\delta_i}{4}\right)^3+\sum_{i=1}^{5}(\alpha-\delta_i)^3$$
$$>5(\alpha+\beta)^3+4(\alpha-\varepsilon)^3+4\left(1-2\beta-\alpha-\frac{5\varepsilon}{4}\right)^3. \tag{14}$$

Therefore, if

$$2(\alpha+\beta)^3>1+\alpha^3+\left(2\beta+\alpha+\frac{5\varepsilon}{4}-1\right)^3, \tag{15}$$

then inequality (9) is violated, and thus the list $\{1,a,a-\varepsilon,-(a+d),-(a+d)\}$ is not realizable. Let $\mathcal{G}$ denote the set of all triples (ε,a,d) that satisfies condition (15). By checking that

$$\sup_{(\varepsilon,a,d)\in\mathcal{G}}\left\{2(\alpha+\beta)^3-\left[1+\alpha^3+\left(2\beta+\alpha+\frac{5\varepsilon}{4}-1\right)^3\right]\right\}=\frac{3}{4}>0, \tag{16}$$

Thus, condition (15) indeed exclude certain symmetrically non-realizable 5-spectra. This gives the following theorem.

Theorem 7

Let $0<\varepsilon,\alpha,\beta$ *satisfy* $\varepsilon<\frac{4+4\sqrt{30}}{29}\alpha$ *and* $0<\varepsilon<1-2\beta<\alpha<1-\beta$. *If* $2(\alpha+\beta)^3>1+\alpha^3+\left(2\beta+\alpha+\frac{5\varepsilon}{4}-1\right)^3$, *then* $\{1,\alpha,\alpha-\varepsilon,-(\alpha+\beta),-(\alpha+\beta)\}$ *is not realizable by any 5-by-5 symmetric nonnegative matrix.*

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4. Conclusion

In conclusion, a class of spectra in the form $\{1, \alpha, \alpha - \varepsilon, -(\alpha + \beta), -(\alpha + \beta)\}$ can be ruled out. Another similar case to consider is spectra of the form $\{1, \alpha, \alpha, -(\alpha + \beta) + \varepsilon, -(\alpha + \beta)\}$. Without a more generic method, trials to extend the same idea to this list should be worthwhile. In addition, higher integer powers of A_i can be used when deducing Theorem 7. This is a way acquire more precision in the constraints on ε and condition (15). Unfortunately, with mere use of this method one can barely terminate the case of $n = 5$ as it can only determine the necessary conditions for a list to be symmetrically realizable.

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