Oscillation and Asymptotic Behavior of a Neutral Dynamic Equation on Time Scales

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Abstract

In this paper, we study the oscillation and asymptotic behavior of a neutral dynamic equation of the form $(x(t)+x(t-r))^{\Delta}+Q(t)G(x(t-\theta))=f(t)$. where $r\ge 0,\theta\ge 0$ are constants, $Q\in C_{rd}(T,\infty), G\in C_{rd}(T,R).$ $xG(x)>0$ for all $x\ne 0, G$ is nondecreasing, $f\in C_{rd}(T,R).$ We obtain some sufficient condition for the oscillation of solutions, as well as some asymptotic behavior is obtained.

Keywords

Oscillation, Neutral, Dynamic Equation,Time Scales

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D.Thesis in 1988 in order to unify continuous and discrete analysis[1]. A time scale *T*, is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications[9].

On any time scale T, we define the forward and backward jump operators by

$$\sigma(t)=\inf\{s>t:s\in T\},\ \rho(t)=\sup\{s<t:s\in T\}.$$

A point $t\in T$, $t>\inf T$, is said to be left-dense if $\rho(t)=t$, right-dense if $t<\sup T$ and $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. The graininess function μ for a time scale T is defined by $\mu(t):=\sigma(t)-t$.

A function $f:T\to R$ is called rd-continuous function provided it is continuous at right-dense points in *T* and its left-sided limits exist (finite) at left-dense points in *T*. The set of rd-continuous functions $f:T\to R$ is denoted by $C_{rd}=C_{rd}(T)=C_{rd}(T,R).$

Let f be a differentiable function on $[a,b]$. Then f is increasing, decreasing,nondecreasing, and non-increasing on $[a,b]$, if $f^{\Delta}(t)>0, f^{\Delta}(t)<0, f^{\Delta}(t)\ge 0,$ and $f^{\Delta}(t)\le 0$ for all $t\in[a,b)$, respectively.

For a function $f:T\to R$ (the range *R* of f may be actually replaced by any Banach space) the delta derivative is defined by

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\sigma(t)-t}, \tag{1}$$

if f is continuous at t and t is right-scattered. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \tag{2}$$

and

$$(\frac{f}{g})^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}; \tag{3}$$

For $t_0, b \in T$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_{t_0}^{b} f^{\Delta}(t)\Delta t = f(b) - f(t_0).$$

An integration by parts formula reads

$$\int_{t_0}^{b} f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_{t_0}^{b} - \int_{t_0}^{b} f^{\Delta}(t)g(t)\Delta t, \tag{4}$$

and infinite integral is defined as

$$\int_{t_0}^{\infty} f(t)\Delta t = \lim_{b\to\infty}\int_{t_0}^{b} f(t)\Delta t. \tag{5}$$

Many differential or difference equations have been widely discussed. For example, Zhu and Li have investigated oscillation on difference equation [14], however,the research on oscillatory solutions of (1) are scarce in the literature.

Our aim in this paper is to obtain the oscillation and asymptotic behavior of the neutral dynamic equation of the form

$$(x(t) + x(t-r))^{\Delta} + Q(t)G(x(t-\theta)) = f(t), \tag{6}$$

where $r \geq 0, \theta \geq 0$ are constants, $Q \in C_{rd}(T,\infty), G \in C_{rd}(T,R)$. $xG(x) > 0$ for all $x \neq 0, G$ is nondecreasing, $f \in C_{rd}(T,R)$.

Definition 1. A solution of (6) is said to be oscillatory if it has arbitrarily large zeros. A solution of (6) is said to be nonoscillatory if it is eventually of constant sign.

Throughout the paper, we assume that the following condition

$$\int_{0}^{\infty} f(t)\Delta t < \infty$$

holds and $\mu = \max\{r, \sigma\}$.

2. Main results

THEOREM 1. Assume that

$$G(x) + G(y) \geq \alpha G(x+y), x>0, y>0,$$

and

$$G(x) + G(y) \leq \beta G(x+y), x>0, y>0,,$$

where $\alpha > 0, \beta > 0$ are constants, and

$$\int_{t_0}^{\infty} Q^{*}(t)\Delta t = \infty,$$

where $Q^{*}(t) = \min\{Q(t), Q(t-r)\}$. Then every solution of (6) oscillates or tends to zero as $t \to \infty$.

Proof:

Let $x(t)$ be a nonoscillatory solution of (6). Therefore there exists large $t_0 \geq 0$ such that $x(t) > 0$ or $x(t) < 0$ for $t \geq t_0$. Without loss of generality, letting $x(t) > 0$, for $t \geq t_0$. Setting

$$z(t) = x(t) + x(t-r),$$
$$\omega(t) = z(t) - F(t),$$

and

$$F(t) = \int_0^t f(s)\Delta s,$$

obviously $z(t) > 0, \omega^{\Delta}(t) = -Q(t)G(x(t-\sigma) \leq 0,$ for $t \geq t_1 > t_0$. Hence $\omega(t) > 0$ or $\omega(t) < 0$,for $t \geq t_2 > t_1$. If $\omega(t) > 0$ for $t \geq t_2$, then $\lim_{t\to\infty} \omega(t)$ exists.

If $\omega(t) < 0$ for $t \geq t_2$, then $0 < x(t) < z(t) < F(t)$ implies that $x(t)$ is bounded and hence $\omega(t)$ is bounded. Thus $\lim_{t\to\infty} \omega(t)$ exists. We claim that $\lim_{t\to\infty} z(t) = 0$. If not, then $z(t) > \lambda > 0$ for $t \geq t_3 > t_2$. From (6), it follows that for $t \geq t_4 > t_3 + 2\mu$.

$$\begin{aligned} f(t) + f(t-r) &= z^{\Delta}(t) + z^{\Delta}(t-r) + Q(t)G(x(t-\theta)) + Q(t-r)G(x(t-r-\theta)) \\ &\geq z^{\Delta}(t) + z^{\Delta}(t-r) + \alpha Q^*(t)(G(x(t-\theta)) + Q(t-r)G(x(t-r-\theta))) \\ &\geq z^{\Delta}(t) + z^{\Delta}(t-r) + Q^*(t)(G(x(t-\theta)) + Q(t-r)G(x(t-r-\theta))) \\ &= z^{\Delta}(t) + z^{\Delta}(t-r) + \alpha Q^*(t)G(z(t-\theta)) \\ &\geq z^{\Delta}(t) + z^{\Delta}(t-r) + \alpha Q^*(t)G(\lambda). \end{aligned}$$

Therefore,

$$z(t_1) < z(t_1) + z(t_1 - r) \leq z(t_4) + z(t_4 - r) - \alpha G(\lambda)\int_{t_4}^{t} Q^*(t)\Delta t + \int_{t_4}^{t} f(t)\Delta t + \int_{t_4}^{t} f(t-r)\Delta t \,,$$

which implies that $z(t) < 0$ for large t. This is a contradiction. Hence the claim holds. Consequently, $\lim_{t\to\infty} x(t) = 0$. Similarly, when $x(t) < 0$ for $t \geq t_0$. We can show that $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

THEOREM 2. Every unbounded solution of Eq.(6) oscillates. In other words, every nonoscillatory solution of Eq.(6) is bounded. The proof is similar to Theorem 1,we omit it here.

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