Adaptive Arithmetic(Research) of Relative Error Best Approximation Beased On Remes Theory

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Abstract

The least square, rectilinear approximation and Newton alternation on curve approximation, their approximation goal is all absolute error. In fact, relative error is usually used but absolute. Relative error approximation according to traditional method is not only difficult but also waste for date. A new best approximation arithmetic for relative error is presented based on Remes and applied on paper quantitative sensor.

Keywords

Relative error, Absolute error, Best approximation adaptation.

1. Introduction

The paper quantitative sensor [1-3] is a precision nonlinear sensor. It can measure accuracy, control accuracy, and directly affect the quality of paper. Therefore, the calibration of the paper quantitative sensor is a very important task. However, it's complicated and difficult to calculate the calibration of the paper quantitative sensor. Most notably, the most critical step is the nonlinear fitting calculation.

Take a paper quantitative sensor as an example, its measurement range is 17.3996 to 171.9462 g/m 2, polynomial order is 3, required relative error is less than 0.5％. In this case, if the absolute error is used as the fitting criterion, the absolute error of the full range should be less than 17.3996×0.5％ =0.086998 g/m 2.

The study found, if fitting algorithm based on absolute error [4-5] in the low range is suitable and accurate, but is too accurate and surplus in other cases. For example, when it's at the high value of the range 171.9462, absolute error of 0.086998 g/m 2 corresponds to a relative error of 0.050596058%. This relative error is ten times higher than the required 0.5%. This is not only unnecessary, but also a serious waste of data resources.

Conversely, if the fit is appropriate at the high value of the range, then at other points, the fit will be insufficient. For example, when it's at the low value of the range 17.3996, absolute error of 0.859731 g/m 2 corresponds to a relative error of 4.9411％. This relative error is ten times lower than the required 0.5%. This fitting result is clearly unacceptable. It would undoubtedly be a very practical algorithm, if it gives curve fitting by polynomial a consistent and minimal relative error over the full scale range (whether at high or low values).

2. General Principles of Cutve Fitting

Generally, $f(x)$ is a continuous function defined on $[a,b]$, $p(x)$ is a polynomial function, There are three general principles for fitting $f(x)$ with $p(x)$:

1). The parameters determination of polynomials $p(x)$ should make the maximum value of the absolute value of the fitting error $Max|f(x)-p(x)|$ minimum, $a \le x \le b$;

2). The parameters determination of polynomials $P(x)$ should make the sum of the absolute values of the fitting errors $\int_a^b |f(x)-p(x)|^2 dx$ minimum;

3). The parameters determination of polynomials P(x) should the sum of squares of absolute values of fitting errors $\int_a^b |f(x)-p(x)|^2 dx$ minimum;

Regardless of Principle 1, Principle 2 or Principle 3, their fitting goals are absolute errors.

3. New Appriximation Algorithm Based on Remes Theory

Remes algorithm [6] is an best infinite approximation algorithms, and its fitting goal is aslo absolute error. But new approximation algorithm given in this paper is, a method which takes relative error as a fitting target, by changing the calculation condition of Remes algorithm, without destroying the convergence and uniqueness of the Remes algorithm. Now we call it weak Remes algorithm.

The calculation processes of weak Remes algorithm:

1), Give n + 2 distinct ordered points arbitrarily,

$$a \le X0 < X1 < X2 \Lambda \ Xi \Lambda \ Xn+2 \le b \quad (1)$$
$$i = 0,1,2,\Lambda \ n+1$$

2), Calculate a0, a1, a2…aj…an and e,

$$f(xi) - p(xi) = (-1)^i ef(xi) \quad (2)$$
$$i = 0,1,2\Lambda \ n+1$$

Among:

$$p(xi) = \sum a_j X^J{}_I$$
$$j = 0,1,2\Lambda \ n$$

3), Find the extremum of the error function

$$e(x) = [f(x) - p(x)]/f(x)$$

4), Compare the extremum of $e(x)$and e value. When the extremum of $e(x)$is greater than the e value, X_I is replaced by the corresponding extremum point of $e(x)$ to form a new ordered series.

5), Determine the convergence accuracy of a_j, X_I and e. When the accuracy requirement is not met, replace the point set in the previous step with a new point set and calculate again. The cycle calculation is finished until the accuracy requirement is met.

4. Programming Principles and Several Details of Weak

Remes algorithm has a very large amount of calculation. If the algorithm is not implemented very well, even if it is running on a computer, it will take a long time.

According to the running function, the program can be divided into five parts Fig. 1.

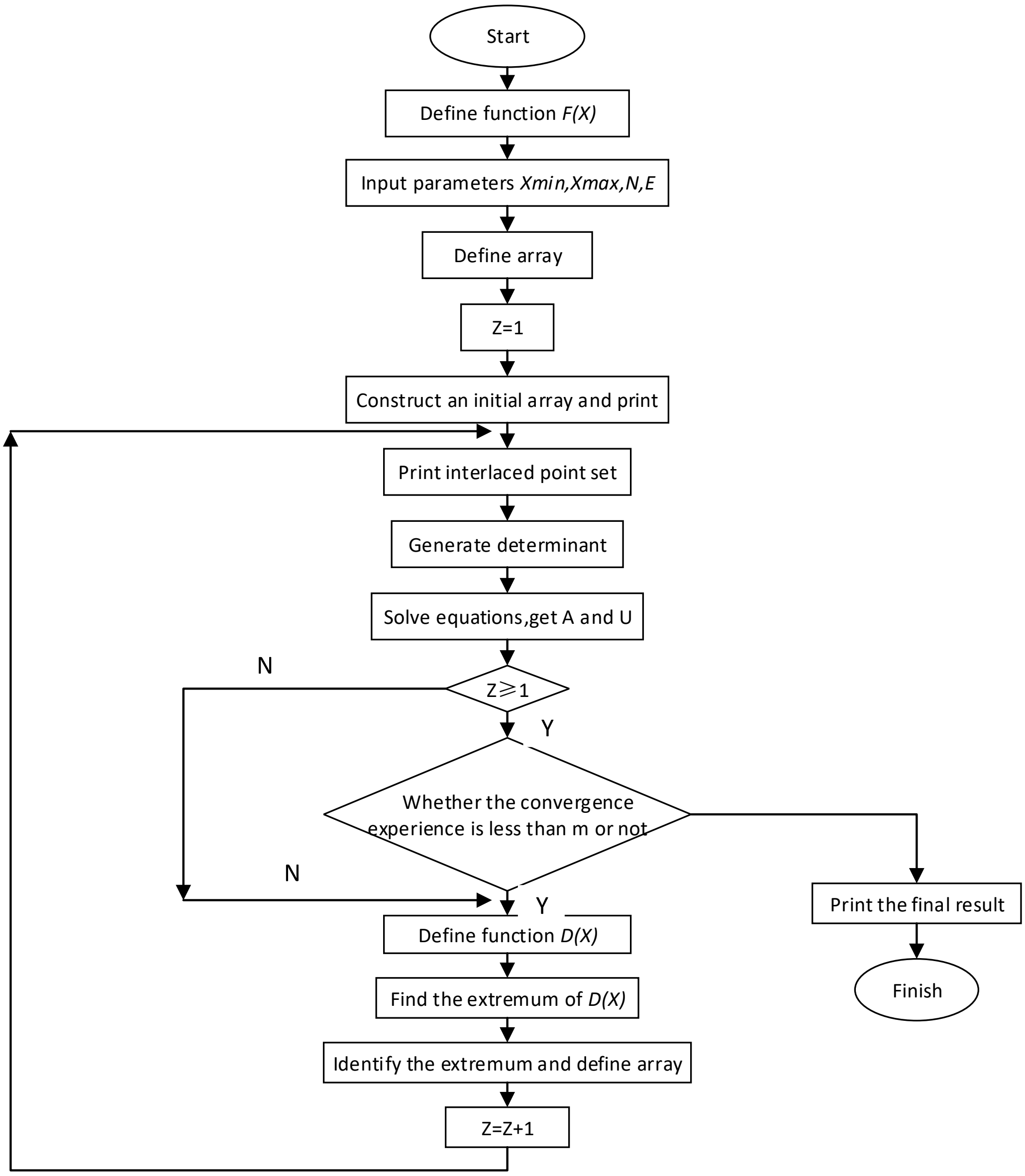

Fig. 1 Block diagram

1), the choice of initial array. If the initial array has good similarity with the Tchebycheff's alternation, it’s undoubtedly effective in saving computing time. We can select the initial array by following the steps below：

After normalization, $F(X)\in C[a,b]$ can be mapped to $F(X)\in C[-1,1]$. And $F'(X)$ is continuous on $[-1,1]$. Based on the above, the Tchebycheff expansion of the function $F(X)$ can be written as,

$$F(X)=\lim_{j\to\infty}S_n(X)=\lim_{j\to\infty}\left(\frac{a_0}{2}\right)+\sum_{j=U}^{n}a_jT_I(X) \tag{3}$$

In this formula,

$$a_j=(2/\pi)\left(\int_{-1}^{1}\left(\left(F(X)T_j(X)\right)\Big/\left(\sqrt{1-X^2}\right)\right)dx\right)$$

$$j=0,1,2,\Lambda\ n$$

$$T_j(X)=Cop\left(j\arccos(X)\right) \qquad -1\le X\le 1$$

$T_j(X)$ is the Chebyshev polynomials[7]. If you take the sum of the first n terms in equation (3) as the approximation polynomial of function $F(X)$, the remaining items are

$$\begin{aligned} R_n(X) &= F(X) - S_n(X) \\ &= a_{n+1}T_{n+1} + a_{n+2}T_{n+2} + \Lambda \\ &= \lim_{j\to\infty}\sum_{I=n+1}^{n} a_j T_j(X) \end{aligned} \tag{4}$$

Because of general conditions, in equation (4), ai converges to zero at a faster rate. So the remainder equation (4) can be written nearly

$$R(X) \approx a_{n+1}T_{n+1}(X) \tag{5}$$

It can be seen from equation (3), $T_{n+1}(X)$ take n+ 2 points on $[-1,1]$

$$\begin{aligned} &X_k = \cos[k\pi/(n+1)] \\ &k = 0,1,2,\Lambda\ n+1 \end{aligned} \tag{6}$$

Intersected the extreme value ± 1 at staggered times and $R_n(X)$ intersects at X_k on the rest of the terms $[-1,1]$ to get plus or minus $\pm a_{n+1}$.

We know from chebyshev's theorem that $S_n(X)$ approximately becomes the best fit polynomial for $F(X)$. The point group obtained according to equation (6) can be mapped to on the interval $[a,b]$:

$$\begin{aligned} &X_k = 0.5\{(b+a)+(b-a)\cos[\pi(n-k+1)/n+1]\} \\ &k = 0,1,2,\Lambda\ n+1 \end{aligned} \tag{7}$$

The set of points determined by formula (7) will approximately become the n times chebychev interleaved point group of $F(X)\in C[a,b]$ on the interval $[a,b]$.

The initial point group of the program is determined according to formula (7). Once the initial point group $\{X_k\}_{k=0} k = n+1$ is determined, the array will be generated by the program:

$$\begin{bmatrix} 1 & X_0 & X_0^2 & X_0^3 & \Lambda & \Lambda & X_0^n & 1 \\ 1 & X_1 & X_1^2 & X_1^3 & \Lambda & \Lambda & X_1^n & -1 \\ 1 & X_2 & X_2^2 & X_2^3 & \Lambda & \Lambda & X_2^n & 1 \\ 1 & X_3 & X_3^2 & X_3^3 & \Lambda & \Lambda & X_3^n & -1 \\ 1 & & M & & M & & & \\ 1 & & M & & M & & & \\ 1 & X_{n+1} & X_{n+1}^2 & X_{n+1}^3 & \Lambda & \Lambda & X_{n+1}^n & (-1)^i \end{bmatrix} \text{and} \begin{bmatrix} F(X_0) \\ F(X_1) \\ F(X_2) \\ F(X_3) \\ M \\ M \\ F(X_{n+1}) \end{bmatrix}$$

Prepare for the solution of the system.

2), Find the extremum.

The maximum point is the cycle of the innermost layer of REMES algorithm[8]. Therefore, the speed and accuracy of the extreme point convergence directly affect the calculation speed and accuracy of the main program. Its calculation method (let's call it folding) is as follows:

① To determine the step length $d = (b-a)/m$, m is an appropriate number between 10 and 100.

② Searching from the starting point to the right by step length d. If $[E(a+d)-E(a)]$ is the same number as $[E(a+(n+1)d)-E(a+nd)]$, then continue searching to the right. Otherwise, write down $a+(n+1)d$ for the return-point. Then, with a step length of $-\left(\frac{1}{10}\right)/d$, search to the right from the reentry point.The search went on and on.

③ In order to improve the convergence speed and precision of extreme points, this program adopts the method of midpoint strengthening approximation. It means:

$$M = [The \quad n \quad return \quad point \ + \ The \quad n-1 \quad return \quad point]/2$$
$$W = [The \quad n-1 \quad return \quad point \ + \ The \quad n-2 \quad return \quad point]/2$$

If $|M - W| < E_1$ (convergence precision of extreme point specified by E1), the $[M + W]/2$ is recorded as the extreme point. Otherwise, search calculations will continue to be performed as instructed ③ .The search went on and on.

After n+2 extreme points are solved, enter the array screening and replacement program.

3), Array screening and replacement procedures.

First of all, we decide whether the extreme value of $D(X)$ at point i is greater than the value of e in the equations (8). If greater than, replace X_i with the extreme value of i, then decide. If $|D(a)| > |D(X_0)|$, replace a with x, otherwise unchanged. If $|D(b)| > |D(X_n + 2)|$, replace b with X_{n+2}, otherwise unchanged.

In some cases, the array will be different. The procedure should be screened. Replace the array at the beginning of the program with the new point group after the above processing, and then re-calculate. Until the convergence accuracy is determined in block 3, and the program stops the loop. Then $P(X)=aX+a_1X^2+\cdots+a_nX^n$, and other auxiliary data results are given.

The program can be written in multiple high-level languages. We programmed this algorithm in Matlab. The operation results show that not only the occupied machine is significantly reduced, but also the calculation precision is very high, which can be easily realized on the current ordinary computer. The program can be used to achieve the best consistent approximation of the relative error of sensor characteristic curve.

5. The Paper Quantitive Sensor Quantitative Curve Calibration Calibration Ecample

5.1 Working principle and calibration requirements of paper quantitative sensor.

A paper quantitative sensor is used to detect the mass or weight per unit area of moving paper. It is a non - contact measurement using the absorption principle of β-ray. The working principle is shown in Fig. 2.

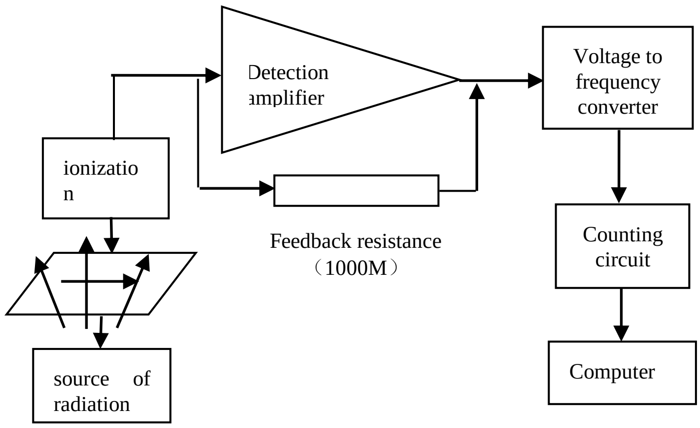

Fig. 2 Sensor schematic diagram

The paper quantitative sensor consists of an β-ray source, a detector and a detection amplifier, a voltage frequency converter, and a pulse counter under the control of the system computer.

Radioactive sources: krypton 85, abundance 40%, 1000 millicurie. Half-life: 10.7 years.

The wave of particles which coming from the β-ray source absorbed by the paper in the air gap. The unabsorbed particles cross the paper and produce a small current ($10^{-9} A$) in the ionization chamber.

The current is a function of the matter between the source and the detector. The amplifier measures the current and converts it into a voltage to a high-precision voltage/frequency converter. The resulting pulse strings are counted by a counter at a precise time interval. This count is the reading of the quantitative sensor, which is a function of paper quantification, but is not a quantitative value. It is read by the computer and converted to paper quantitative values by the scale equation stored in the computer. At the same time dust, air gap, temperature, source and receiver air column temperature compensation.

The scale equation is different with the variety of paper tested or the change of production conditions. Under certain conditions, the corresponding scale equation is generated by calculating the quantitative experimental data of a certain paper, that is, calibrating the paper quantitative sensor.

There are strict regulations on the calibration precision in the industry, and the general requirements are less than 0.5%.

5.2 Calibration of paper quantitative sensor.

The quantitative measurement is carried out by the method of scale:

$$Ratio = \frac{Measuring \quad the \quad count \quad - \quad background}{Air \quad count \quad - \quad background}$$

Background: Used to correct amplifier drift, natural radiation effects, generally constant;

“Air” count: The count of only air in the measured air gap. Depending on the magnification, source strength, air gap temperature and dust accumulation to compensate for the effects of dust;

“Flag” count: It is the count of placing flag on the β-ray source channel.

“Measuring” count: The "sample paper" (external standard), "flag" and the paper in the production process of measurement reading are "measurement" counts.

“Ratio” Measuring the ratio: Software will "empty" count, "measurement" count minus "background" respectively, calculated the ratio. The uncorrected quantitative value is calculated from this ratio. There is an approximate exponential curve between "Ratio" and quantitative value. The curve in the software is divided into several sections, which are fitted with

$$BW = BA_0 + BA_1 X + BA_2 X^2 + BA_3 X^3 \tag{9}$$

in the quantitative value interval of each section.

X is the Ratio which called In(R).

BA0-BA3 is the calibration coefficient, which depends on the paper species and other factors.

The data of a certain paper species are as follows:

Table 1 The data of a certain paper species

Serial number	R values	X(lnR)	Yi The experiment of quantitativeg/m2
1	0.954918	-0.04613	17.398
2	0.912041	-0.09207	34.699
3	0.870559	-0.13862	51.916
4	0.830108	-0.18620	69.262
5	0.791314	-0.23406	86.498
6	0.753557	-0.28295	103.771
7	0.717703	-0.33170	120.875

8	0.682713	-0.38168	138.068
9	0.649443	-0.43164	155.128
10	0.618066	-0.48116	171.744

Construct original function.

Ten R values are obtained from ten standard flag boards, and ten X values are converted from ten R values. The 10 X value corresponds to the calibration ration of 10 laboratories. Since the production is continuous, although the calibration laboratory can only give ten sets of X and Yi values, it is indisputable that the original function formed by X and Yi is continuous function.

According to the the weierstrass theorem, its original function can be expressed by the following polynomial:

$$Fne(X)=a_0+a_1X+a_2X_2+a_2X_2+\Lambda\ \Lambda\ +a_9X_9 \tag{10}$$

By solving the system of equations by computer, the following equation can be obtained:

a0=1.301008565303×101

a1=4.040819693776×102

a2=1.869280980042×104

a3=2.376022482500×105

a4=1.791897295330×106

a5=8.432595653996×107

a6=2.497618428792×107

a7=4.520725715560×107

a8=4.563332341189×107

a9=1.966606115071×107

The construction of the original function, which makes full and necessary use of all ten sets of original data, shows that the construction of the original function is reasonable.

According to the weierstrass theorem, his fitting error can be estimated as:

$$\varepsilon < Fne(\xi)^{(n)} / X < 1\times10^{-29}$$

This is much higher than the 0.5% required by the system, which indicates that the original function constructed is reliable.

Build mathematical models.

The mathematical model required by the control system is:

$$BW = BA_0 + BA_1X + BA_2X_3 + BA_3X_3 \tag{11}$$

The weak REMES algorithm presented in this paper is applied, taking equation (10) as the original function and equation (10) as the mathematical model.The fitting results are as follows:

BA0=-1.03204069015582×10-2

BA1=-3.78813930602975×10+2

BA2=-2.00207491433175×10+1

BA3=+5.59020100202040×10+1

The fitting error is shown in Table 2.

Table 2 The fitting error

Serial number	R values	maximum error X(lnR)	Maximum relative error μ%	Absolute error g/m2
1	0.954918	-0.04613	+0.4288348	+0.074609
2	0.561788	-0.57663	-0.4288348	-0.092502
3	0.905273	-0.09952	+0.4288348	+0.160828
4	0.748143	-0.29016	-0.4288348	-0.455880
5	0.625591	-0.46909	+0.4288348	+0.720999

From Table 2, it can be seen that the maximum relative error of the fitting of equation (11) and equation (10) of the mathematical model is $|\mu_{\max}| = 0.4288348\%$. The mathematical model formula (11) intersects $\pm|\mu_{\max}|$ at points -0.04613, -0.57663, -0.09952, -0.29016 and -0.46909.

This is the optimal fitting of mathematical model formula (11) and functional formula (10) with the objective of relative error. In other words, fitting the original function equation (10) with the mathematical model formula (11) will not make the relative error greater than $|\mu_{\max}|$ in any case. $|\mu_{\max}|$ is the best fitting error, and equation (11) is the best fitting formula.

In fact, the fitting error of the above mentioned fitting is much larger than that of $|\mu_{\max}|$ when the least square method and REMES and other traditional algorithms are applied. Unable to meet system requirements.

Error comparison.

We compared the fitting errors of the four methods of weak REMES, REMES, least squares and data provided by foreign companies. The results of the comparison are as follows:

Table 3 Coefficient comparison table

	Weak REMES algorithm%	Algorithms provided abroad%	REMES algorithm%	Least square method%
BA0	-0.010	-0.240	+0.191	-0.1997
BA1	-378.814	-385.566	-376.981	-384.746
BA2	-20.021	-69.968	-36.752	-66.053
BA3	+55.902	-24.163	+9.798	-19.072

Table 4 Relative error comparison table

Serial number	Weak REMES algorithm%	Algorithms provided abroad%	REMES algorithm%	Least square method%
1	+0.428835	-0.016276	-0.136594	-0.007676
2	-0.428835	+0.264383	+0.140814	+0.267890
3	+0.428835	-0.054114	-0.158089	-0.062183
4	-0.428835	+0.046198	+0.311413	+0.034600
5	+0.428835	-0.029482	+0.308721	-0.035818
6		+0.023157	+0.501829	+0.028646

7		-0.069912	-1.129436	-0.050915
8		+0.126902		+0.148615
9		-0.900886		-0.924773

It can be seen from tab-4 that the fitting results obtained by foreign algorithms, REMES algorithm and least square method cannot meet the requirement of the control system for the relative error less than 0.5%.Among them, there is an error in the algorithm provided abroad, near X=-0.57677.REMES algorithm has two out-of-tolerance, in the vicinity of X=-0.11058 and X=-0.05447 respectively. The least square method has a super difference, respectively, near X=-0.05737.Only the maximum relative error of the weak REMES algorithm fitting is + 0.428835% < 0.5%, which fully meets the requirements.

6. Conclusion

1. The polynomial relative error approximation curve of any continuous curve in the real number field is not only present but unique;

2. The optimal approximation curve of the polynomial cannot be accurately obtained by theoretical derivation of finite steps, but it can be obtained by finite cycle calculation under the precondition of specifying fitting precision and fitting order.

3. The weak REMES algorithm not only converges, but in general, converges faster than the one with the best approximation of absolute error.

This algorithm has the advantages of high calculation accuracy and wide application range. It can not only be used for curve fitting calculation, but also for experimental data processing, establishing mathematical model for computer control system, or generating elementary functions of computer software. This algorithm has been used in QCS system of imported paper machine and calibration calculation of paper quantitative sensor.

The disadvantage of the algorithm is that the calculation principle is complex and the program is well compiled.

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