The Analytical Solution of Deformation Mode of Crack Beam Under Pulse Loads

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Abstract

Cantilever beam with crack under the pulse load can be considered to be an common dynamics behavior in structural engineering, in which the deformation mode of crack beam (CB) caused by pulse load can indicate their important properties in the strength analysis of structures. Using rigid plastic model of cantilever beam materials, this paper studies a sort of defect, and reveals the adhesive movement of movable hinge for such CB under strong impact load on account of crack effect, which is rather different with crack-free beam. Standing on the assumption of centralization and localization of crack effect, the coupling and second-order effect are ignored in the simulation procedure of the deformation mode of CB. In spite of analysis results of the deformation mode of CB are theoretically approximate to some degree, it is quite practical and useful in the assessment of structural safety.

Keywords

Crack beam (CB); pulse load; cantilever; deflection model; plastic hinge.

1. Introduction

The analysis of deformation mode of beam member structure under pulse loads have been paid great attention globally. Early in 1952, Lee and Symonds (1952) analyzed the rigid-plastic dynamic response of straight beam with two free ends under triangle pulse that is imposed on the mid-strut, which subsequently demonstrated that the free beam was forced to move in rigid body by small loads. However, one plastic hinge appears in the mid-strut and the other two plastic hinge in its both sides when $PL/M_p > 4$ (P maximum value of loads, L beam span and M_p yield moment of beam). Meanwhile the plastic hinges in two sides of the beam move to the middle as PL/M_p increases, which could lead to a new fundamental concept: Moving Hinge. Parkers' model was built up by Parkes in 1955, who studied the case that a cantilever beam is hit by a rigid block (with mass G and initial velocity V_0) and its dynamic response under the premise of bonding the rigid block to the beam after hitting. Stronge (1990) and Yu (1990) illustrated that a shrinking plastic loading area and expanding unloading area appear in the beam after hitting between a rigid-linear hardening cantilever beam and a rigid block. The interface of the both areas moves toward the root of beam and was changed into the Parkes' moving hinge when the material of beam hardens slightly.

This paper analyzes the deformation mode of defective cantilever beam in which there is one of the most common type of cracked defect in practical structures. As the crack is a structural singularity of cantilever beam, it will significantly discount the bending moment of plastic yield at the cracked section and dramatically change the distribution of stress and strain in CB, which is quite different deformation mode than crack-free beam.

We check the positions and moving of plastic hinge of beam-type members considering bending effects in different level, displacements, velocity and accelerate speed of different sections, distribution of bending moment, shearing force of the sections in rigid zone at different time. In the paper, we analyzed a criterion whether cracked-beam-type member could overlook influence imposed by the defects under the suddenly applied loads. Meanwhile, on the premise of considering influence of defects, conditions, positions of plastic hinge and basic relation of deformation modes in different segments were studied in beam type members. Consequently, we uncovered the influence on the shock resistance imposed by cracks existed in beams. Standing on some fundamental assumption and ignoring second-order effects, these conclusions are useful and valuable for the same problems.

2. Stress Field Around the Cracks

Based on the fundamental assumption of elastics, we arrive at the governing equations on the stress distribution near the cracks. Inappropriate partly for the case of rigid-plastic beam, prerequisite for the establishment will be modified upon for the rigid-plastic beam.

2.1 Stress and Strain Field Near I-Type Crack Tips

A central through-wall crack with length 2a cutting through a two-dimension infinite plate, the two-way tensile stresses are imposed at infinite as shown in Figure 2.1. Subsequently, Z-type in the Westergaard stress function could be confirmed through corresponding boundary conditions.

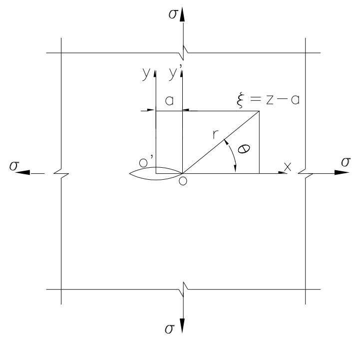


Fig. 2.1 Cracks model

For I-type cracks under both-way tensile stress, boundary conditions can be described as following:

(1) $\sigma_y = 0$ on $y = 0\,, -a \le x \le a$. For unstressed cavity exists in cracks, the all components of stress are 0.

(2) $\sigma_y > \sigma$ on $y = 0\,, |x| > a$,the closer to a for x, the bigger σ_y. For stress concentration happens on the front of cracks, furthermore, the closer to the crack tip, the higher degree of concentration.

(3) $\sigma_y = \sigma,\ \sigma_x = \sigma$ on $y = 0\,, x \to \pm\infty$. For the position farther from cracks, the less degree of stress concentration.

According to the above boundary conditions, the Westergaard stress function can be read as

$$z(x, y) = \frac{\sigma z}{\sqrt{z^2 - a^2}} \tag{1}$$

If origin of coordinates is moved from centre to the crack tip, $\xi(r, \theta) = r cos\theta + i r sin\theta = z - a$ in the form of polar coordinates. Further, on $r \ll a\,, r \to 0$ we can obtain stress distribution below

$\xi(r, \theta) \to 0$ for $r \to 0$

then, $z = \xi + a \to a$ for $z + a \to a$

$$z(x, y) = \frac{\sigma z}{\sqrt{z^2 - a^2}} = \frac{\sigma z}{\sqrt{2a\xi}} = \sigma\sqrt{\frac{a}{2}}\, r^{-\frac{1}{2}} e^{-i\frac{\theta}{2}} \tag{2}$$

and its trigonometric function form can be written as $z = \sigma\sqrt{\frac{a}{2r}} cos\frac{\theta}{2} + i\left(-\sigma\sqrt{\frac{a}{2r}} sin\frac{\theta}{2}\right)$

So we get the Westergaard stress function as following

$$\sigma_x = -\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{3}$$

$$\sigma_y = -\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{4}$$

$$\tau_{xy} = -\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (5)$$

where $K_I = \sigma\sqrt{\pi a}$ is stress intensity factor of I-type crack.

2.2 Stress Distribution of Crack Tip Considering Plastic Evolution

Under the assumption of linear elasticity, all principal stresses surpass material yielding stress in the above situation, which is not in keeping with the assumption of rigid plastic, stress redistribution will be triggered in the area that stress exceeds yield limit and yield region will be enlarged which could not be ignored consequently. A new plastic region should be checked.

According to equilibrium of forces in infinite body, the sum of stress in the x direction should be equal when the stress redistribution takes place. As is shown in Figure 2.2, the integration area of curve EBF equals to that of ABCD. Meanwhile, areas under BF and CD are identical for the sake of infinite body. Here we formulate

$$r \cdot \sigma_s = \int_0^{r_0} \sigma_y(r)\,\mathrm{d}r \quad (6)$$

where r is modified curve in plastic region and σ_s yielding stress of CB. Assuming $\sigma_r = \frac{K(\theta)}{\sqrt{r}}$ we could yield $K(\theta)\cdot 2\,r^{\frac{1}{2}}\Big|_0^{r_0} = r\cdot\sigma_s$, and lead to $r = 2r_0$.

$$K^2(\theta) = \frac{K^2}{2\pi},\ r_0 = \frac{K^2}{2\pi\sigma_s^2},\quad r = \frac{K^2}{\pi\sigma_s^2}\ \text{for}\ \theta = 0 \quad \sigma_r = \frac{K(\theta)}{\sqrt{r}} \quad (7)$$

in which K is stress intensity factor.

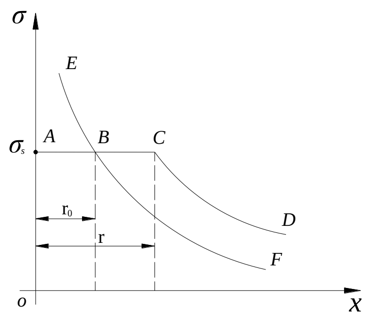


Fig. 2.2 Correction of plastic range

For $\frac{K^2}{\pi\sigma_s^2} < \omega = \frac{H}{2} - \alpha$ (i.e. $K > \sigma_s\sqrt{\pi\left(\frac{H}{2} - a\right)}$) , where H is height of beam, a half height of crack, The whole cross section of beam has not yet turned into plastic range and the cracks make no difference to rigid-plastic distribution of whole beam, which means that the influence imposed by cracks could be overlooked here under impact loads.

3. Deformation Pattern of CB Under Pulse Loads

It is necessary to keep a watchful eye on the influence of defects on beam structure under pulse loads if $K > \sigma_s\sqrt{\pi\left(\frac{H}{2} - a\right)}$. So we study the situation only if $K > \sigma_s\sqrt{\pi\left(\frac{H}{2} - a\right)}$ below, in which we adopt the simplified prerequisites that the cracks have the localized effect on beam and the cracks are concentrated as a plastic hinge when the beam turns into plastic range near the crack.

Parkes modified model is depicted in Figure 3.1 for the case $K > \sigma_s\sqrt{\pi\left(\frac{H}{2} - a\right)}$. A plastic hinge emerges at the point H of the beam when a heavy load $p_0 \geq p_s$ (p_s plastic yielding load) is suddenly applied on CB. There are two different zones of deformation, plastic and rigid zones, appeared in CB. When the crack is in the plastic zone, the presence of crack has not effect on the deformation of beam for the reason of whole cross section of beam is in the phase of plastic deformation. If not, a new deformation pattern (double hinge pattern) will be generated.

When $K > \sigma_s\sqrt{\pi\left(\frac{H}{2} - a\right)}$, we make some simplification in order to rule out indecent factors and discover substantive characteristics of component responses. On account that length of the beam surpasses largely that of plastic extension caused by cracks, it is a feasible way that the crack of beam is localized and replaced as a plastic hinge.

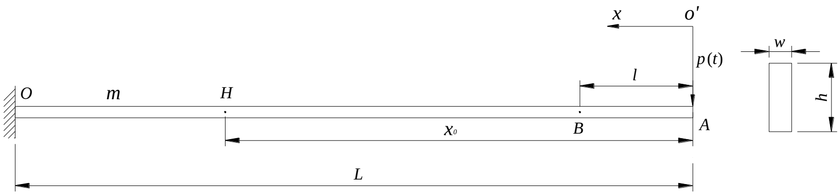


Fig. 3.1 Parkes modified model

The function of pulse loads can be written as

$$p = \begin{cases} p_0 & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases} \tag{7}$$

in which τ is the duration of pulse load, as showed in Figure 3.2.

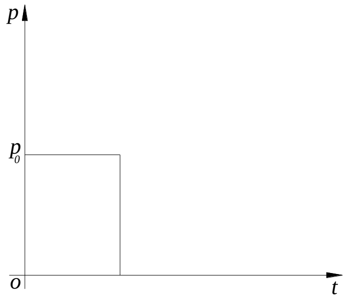


Figure 3.2 pulse load

The cantilever beam and crack model are shown in Figure 3.1 and 3.3 respectively. When $K > \sigma_S\sqrt{\pi(\frac{H}{2} - a)}$, the beam near crack B is in plastic state and the normal stress equals to σ_S on whole cross section of crack where the moment and shear force are denoted by M_B and Q_B. In all phases of deformation of CB under different load level and at different time, the deformation modes of CB are depicted as following.

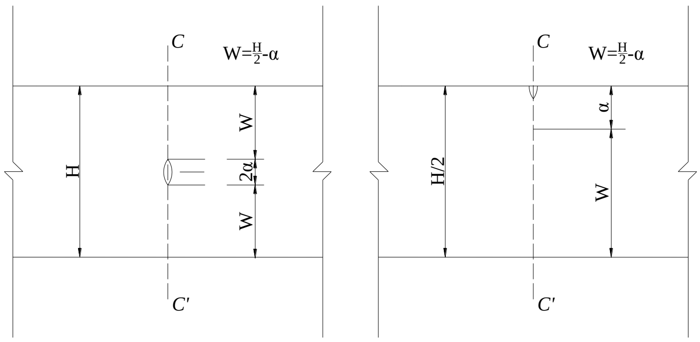


Figure 3.3 Crack model

3.1 If $p_0 < 3\frac{M_B}{x_B}$ and $3\frac{M_s - M_B}{Q_B} < L$

(a) For $0 \leq t \leq \tau$

The motion pattern of cantilever beam is identical to that of suddenly applied load which terminates at $t = \tau$. At the beginning time t=0, the momentum variation in the cantilever beam can be written as

$$\frac{1}{2} m x_B \cdot \frac{3t}{m x_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{1}{2} m \cdot 3\,\frac{M_s - M_B}{Q_B} \cdot \frac{2Q_B^2 t}{3m(M_s - M_B)}\bigg|_{t=\tau} = p_0\tau \tag{8}$$

and variation of angular momentum is

$$\frac{1}{6}mx_B^2\cdot\frac{3t}{mx_B}\left(p_0-\frac{M_B}{x_B}\right)+\frac{1}{6}m\cdot 9\,\frac{(M_s-M_B)^2}{Q_B^2}\cdot\frac{2Q_B^2t}{3m(M_s-M_B)}\bigg|_{t=\tau}=\frac{\tau}{2}(\,p_0x_B-3M_B+2M_s)\quad(3.1.3)$$

During this phase there are two plastic hinges in the CB, one of which lies on the crack, and another occurs at the location $x_0=3\frac{M_S-M_B}{Q_B}$. Both are stationary plastic hinge. The displacement of free end are

$$z(t)=\frac{3t^2}{2mx_B}\left(p_0-\frac{M_B}{x_B}\right)+\frac{Q_B^2t^2}{3m(M_S-M_B)}\quad(9)$$

We can obtain the angle of AH

$$\theta(t)=\frac{3t^2}{2mx_B^2}\left(p_0-\frac{M_B}{x_B}\right)+\frac{Q_B^3t^2}{9m(M_S-M_B)^2}\quad(10)$$

When $t=\tau$, then

$$\theta_\tau=\frac{3\tau^2}{2mx_B^2}\left(p_0-\frac{M_B}{x_B}\right)+\frac{Q_B^3\tau^2}{9m(M_S-M_B)^2}\quad(11)$$

$$Z_\tau=\frac{3\tau^2}{2mx_B}\left(p_0-\frac{M_B}{x_B}\right)+\frac{Q_B^2\tau^2}{3m(M_S-M_B)}\quad(12)$$

(b) For $\tau\le t\le t_0$

Because of no applied loads during this phase, there are not the singular distribution of plastic zones and the motion mode of whole beam is identical to the situation of crack-free beam, which has only one plastic hinge. For the reason of the time boundary conditions in the CB are different from the one in crack-free beam, the unique features of CB occur during this phase.

The conservation law of momentum and angular momentum of CB are depicted as following

$$\begin{cases}\frac{1}{2}mx_0(t)\dot z|_t-\frac{1}{2}mx_0(t)\dot z|_{t=\tau}=\int_\tau^t p(t)\mathrm{d}t\\ \frac{1}{6}mx_0^2(t)\dot z|_t-\frac{1}{6}mx_0^2(t)\dot z|_{t=\tau}=M_s(t-\tau)\end{cases}\quad(13)$$

then

$$x_0(t)=\frac{3M_St+\frac{3\tau}{2}(p_0x_B-3M_B)}{p_0\tau}\qquad \dot z(t)=\frac{2p_0^2\tau^2}{3m\left[M_St+\frac{\tau}{2}(p_0x_B-3M_B)\right]}\quad(14)$$

By the above equations, it is easy to get

$$x_0(\tau)=\frac{3}{2}\left(x_B+\frac{2M_S-3M_B}{p_0}\right)\quad \text{for}\quad t=\tau\quad(15)$$

For generality $x_0(\tau)\ne x_B$, $x_0(\tau)\ne x_0=3\frac{M_S-M_B}{Q_B}$, it means that there is a suddenly change of position of plastic hinge at ending moment of loading.

In formula mentioned above, t_0 means the moment when $x(t)=L+x_B$, so

$$\frac{3}{2}\left(x_B-3\frac{M_B}{p_0}\right)+\frac{3}{p_0}\cdot\frac{t_0}{\tau}=L+x_B\quad(16)$$

Then we get

$$t_0=\frac{p_0\tau}{3M_S}\left(L+\frac{9}{2}\frac{M_B}{p_0}-\frac{1}{2}x_B\right)\quad(17)$$

Now the speed of free end A can be written as

$$\dot z(t_0)=\frac{2p_0\tau}{m(L+x_B)}\quad(18)$$

while angular velocity

$$\alpha(t_0)=\frac{2p_0\tau}{m(L+x_B)^2}\quad(19)$$

and angular acceleration

$$\omega(t_0)=-\frac{12M_S}{m(L+x_B)^3}\quad(20)$$

In this phase we can get that

$$x_0(t)=3\frac{M_St+\frac{\tau}{2}(p_0x_B-3M_B)}{p_0\tau}=\frac{3}{2}\left(x_B-3\frac{M_B}{p_0}\right)+\frac{3M_S}{p_0}\cdot\frac{t}{\tau}\quad(21)$$

which means plastic hinge here can be regarded as migration hinge and its motion speed is

$$v_H = \frac{dx(t)}{dt} = \frac{3M_S}{p_0} \cdot \frac{1}{\tau} \tag{22}$$

From this, we can get

$$z(t) = \frac{2p_0^2\tau^2}{3mM_S}\left\{\ln\left[M_S t + \frac{\tau}{2}(p_0 x_B - 3M_B)\right] - \ln\left[M_S\tau + \frac{\tau}{2}(p_0 x_B - 3M_B)\right]\right\} + z_\tau \tag{23}$$

$$= \frac{2p_0^2\tau^2}{3mM_S}\left\{\ln\left[M_S t + \frac{\tau}{2}(p_0 x_B - 3M_B)\right] - \ln\left[M_S\tau + \frac{\tau}{2}(p_0 x_B - 3M_B)\right]\right\} + \frac{3\tau^2}{2mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3 t^2}{3m(M_S - M_B)}$$

We arrive at the angle of AH

$$\theta(t) = \frac{2p_0^3\tau^3}{9mM_S}\left(\frac{1}{M_S\tau + \frac{\tau}{2}(p_0 x_B - 3M_B)} - \frac{1}{M_S t + \frac{\tau}{2}(p_0 x_B - 3M_B)}\right) + \theta_\tau \tag{24}$$

$$= \frac{2p_0^3\tau^3}{9mM_S}\left(\frac{1}{M_S\tau + \frac{\tau}{2}(p_0 x_B - 3M_B)} - \frac{1}{M_S t + \frac{\tau}{2}(p_0 x_B - 3M_B)}\right) + \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3\tau^2}{9m(M_S - M_B)^2}$$

When $t = t_0 = \frac{p_0\tau}{3M_S}\left(L + \frac{9}{2}\frac{M_B}{p_0} - \frac{1}{2}x_B\right)$, then

$$\theta_0 = \frac{p_0^2\tau^2}{3mM_S}\left(\frac{1}{x_0(\tau)} - \frac{1}{L+x_B}\right) + \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3\tau^2}{9m(M_S-M_B)^2} \tag{25}$$

$$Z_0 = \frac{2p_0^2\tau^2}{3mM_S}\ \ln\frac{L+x_B}{x_0(\tau)} + \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3\tau^2}{9m(M_S-M_B)^2} \tag{26}$$

(c) For $t_0 \le t \le t_1$

There is no external loads on beam this moment and plastic hinge exists on the root. The solutions of motion equations are presented below,

$$x(t) = L + x_B \tag{27}$$

$$\dot{z}(t) = \frac{3}{m(L+x_B)}\left(p_0\tau - p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right) \tag{28}$$

$$\alpha(t) = \frac{3}{m(L+x_B)^2}\left(p_0\tau - p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right) \tag{29}$$

The motion of CB terminates when

$$\alpha(t) = \frac{3}{m(L+x_B)^2}\left(p_0\tau - p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right) = 0 \tag{30}$$

which we mark this moment as t_1. And motions of whole beam stop when $t = t_1 = t_0 + \frac{2p_0}{3p_S}\tau$.

When $t_0 \le t \le t_1 = \left(\frac{p_0}{p_S} + \frac{3}{2}\cdot\frac{M_B}{M_S} - \frac{1}{2}\cdot\frac{p_0}{p_S}\cdot\frac{x_B}{L+x_B}\right)\tau$, a stationary plastic hinge at this moment locates at the fixed end. From $\dot{z}(t) = \frac{2p_0\tau}{m(L+x_B)} - \frac{3p_0(t-t_0)}{m(L+x_B)} = \frac{3}{m(L+x_B)}\left(p_0\tau - p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right)$ and $\dot{\theta}(t) = \alpha(t) = \frac{3t}{m(L+x_B)^2}\left(p_0\tau - p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right)$ we can obtain the speed of free end A

$$z(t) = \frac{3t}{m(L+x_B)}\left(p_0\tau - \frac{1}{2}p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right) + z_0 \tag{31}$$

and the angle of AO, respectively

$$\theta(t) = \frac{3t}{m(L+x_B)^2}\left(p_0\tau - \frac{1}{2}p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right) + \theta_0 \tag{32}$$

At the time $t_1 = \left(\frac{p_0}{p_S} + \frac{3}{2}\cdot\frac{M_B}{M_S} - \frac{1}{2}\cdot\frac{p_0}{p_S}\cdot\frac{x_B}{L+x_B}\right)\tau$, the movement of CB terminates and its maximum angle and displacement are respectively,

$$\theta_1 = \frac{3p_S t_1^2}{2m(L+x_B)^2} + \frac{p_0^2\tau^2}{3mM_S}\left(\frac{1}{x_0(\tau)} - \frac{1}{L+x_B}\right) + \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3\tau^2}{9m(M_S-M_B)^2} \tag{33}$$

$$Z_1 = \frac{3p_S t_1^2}{2m(L+x_B)} + \frac{2p_0^2\tau^2}{3mM_S}\ln\frac{L+x_B}{x_0(\tau)} + \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3\tau^2}{9m(M_S-M_B)^2} \tag{34}$$

3.2 For $p_0 < 3\frac{M_B}{x_B}$ and $\frac{M_S - M_B}{Q_B} \le L \le 3\frac{M_S - M_B}{Q_B}$

(a) When $0 \le t \le \tau$

At this stage, motion pattern of CB under suddenly applied loads is the same as the former. For the beam segment AB, its linear and angular displacement are following,

$$\dot{z}(t) = \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) \tag{35}$$

$$\alpha(t) = \frac{3t}{mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) \tag{36}$$

and for the beam segment BO

$$\dot{z}(t) = \frac{3[Q_B L - (M_S - M_B)]}{mL^2} t \tag{37}$$

$$\alpha(t) = \frac{3[Q_B L - (M_S - M_B)]}{mL^3} t \tag{38}$$

Compared with the initial time, variation of the momentum and angular momentum in whole beam at $t = \tau$ are as follows, respectively.

$$\frac{1}{2} mx_B \cdot \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{1}{2} mL \cdot \frac{3[Q_B L - (M_S - M_B)]t}{mL^2}\bigg|_{t=\tau} = \frac{3\tau}{4}\left(p_0 + \frac{M_B}{x_B} - 2\frac{M_S - M_B}{L}\right) \tag{39}$$

$$\frac{1}{6} mx_B^2 \cdot \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{1}{6} mL^2 \cdot \frac{3[Q_B L - (M_S - M_B)]t}{mL^2}\bigg|_{t=\tau} = \frac{\tau}{4}\left(3M_B \frac{L}{x_B} + 2p_0 x_B - p_0 L - 2M_S\right) \tag{40}$$

There are two plastic hinges on CB and both are stationary hinges, one of which lies on its cracks and another on its root.

The velocity of free end can be written as:

$$\dot{z}(t) = \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{3[Q_B L - (M_S - M_B)]}{mL^2} t \tag{41}$$

From

$$\dot{\theta}(t) = \alpha(t) = \frac{3t}{mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{3[Q_B L - (M_S - M_B)]}{mL^3} t$$

we gain the angle of the segment AH:

$$\theta(t) = \frac{3t^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{3[Q_B L - (M_S - M_B)]}{2mL^3} t^2 \tag{42}$$

If $t = \tau$, $\theta_\tau = \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{3[Q_B L - (M_S - M_B)]}{2mL^3}\tau^2$, we obtain the following

$$x_0(t) = \frac{3\tau^2}{2mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{3[Q_B L - (M_S - M_B)]}{2mL^2}\tau^2 \tag{43}$$

(b) When $\tau \le t \le t_0$

For the situation of without external loads at this time, there is not singular plastic area in CB, with the case identical as crack-free beam, so it is the single hinge mode. However, for the sake of differences in temporal conditions, the CB presents the unique features, which are few differences from crack-free beam at the stage, and are showed as follows.

$$\begin{cases} \frac{1}{2} mx_0(t)\dot{z}\Big|_{t=t} - \frac{1}{2} mx_0(t)\dot{z}\Big|_{t=\tau} = \int_\tau^t P(t)dt \\ \frac{1}{6} mx_0^2(t)\dot{z}\Big|_{t=t} - \frac{1}{6} mx_0^2(t)\dot{z}\Big|_{t=\tau} = M_S(t - \tau) \end{cases} \tag{44}$$

Then we can arrive at

$$\begin{cases} \frac{1}{2} mx_0(t) \cdot \dot{z} = \frac{3\tau}{4}\left(p_0 + \frac{M_B}{x_B} - 2\frac{M_S - M_B}{L}\right) \\ \frac{1}{6} mx_0^2(t)\dot{z} = M_S(t - \tau) + \frac{\tau}{4}\left(3M_B \frac{L}{x_B} + 2p_0 x_B - p_0 L - 2M_S\right) \end{cases} \tag{45}$$

and

$$x_0(t) = \frac{4M_S t + \tau\left(3M_B \frac{L}{x_B} + 2p_0 x_B - p_0 L - 6M_S\right)}{\tau\left(p_0 + \frac{M_B}{x_B} - 2\frac{M_S - M_B}{L}\right)} \tag{46}$$

$$\dot{z}(t) = \frac{3\tau^2}{2m} \frac{\left(p_0 + \frac{M_B}{x_B} - 2\frac{M_S - M_B}{L}\right)^2}{4M_S t + \tau\left(3M_B \frac{L}{x_B} + 2p_0 x_B - p_0 L - 6M_S\right)} \tag{47}$$

$$\alpha(t)=\frac{3\tau^3}{2m}\frac{\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^3}{\left[4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)\right]^2} \quad (48)$$

$$\omega(t)=-\frac{12\tau^3}{m}\frac{\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^3M_S}{\left[4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)\right]^3} \quad (49)$$

As per eq. (46) , when $t=\tau$ then $x_0(\tau)=\frac{3M_B\frac{L}{x_B}+2p_0x_B-p_0L-2M_S}{p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}}$ for $x_0(\tau)\neq x_B$. In general, $x_0(\tau)\neq x_0=L$, which means that the position of plastic hinge will suddenly change at the moment of loads ending. In which t_0 is the time when $x(t)=L+x_B$. For $\frac{4M_St_0+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)}{\tau\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)}= L+x_B$, we can get

$$t_0=\left[\left(1-\frac{x_B}{2L}\right)+\frac{M_B}{4M_S}\left(3+4\frac{L}{x_B}+2\frac{x_B}{L}\right)+\frac{p_0L}{4M_S}\left(2-\frac{x_B}{L}\right)\right]\tau \quad (50)$$

Now, the speed of free end A is:

$$\dot{z}(t_0)=\frac{3\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)\tau}{2m(L+x_B)} \quad (51)$$

and angular velocity of the beam:

$$\alpha(t_0)=\frac{3\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)\tau}{2m(L+x_B)^2} \quad (52)$$

and angular acceleration of the beam:

$$\omega(t_0)=\frac{12M_S}{m(L+x_B)^3} \quad (53)$$

$$x_0(t)=\frac{4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)}{\tau\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)}=\frac{3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S}{p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}}+\frac{4M_S}{p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}}\cdot\frac{t}{\tau}$$

Which illustrates plastic hinge here is migration hinge, and its speed, $v_H=\frac{dx(t)}{dt}=\frac{4M_S}{p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}}\cdot\frac{1}{\tau}$, is constant obviously.

Note that $\dot{z}(t)=\frac{3\tau^2}{2m}\frac{\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^2}{4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)}$, we can arrive at this

$$Z(t)=\frac{3\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^2\tau^2}{8mM_S}\left\{\ln\left[4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)\right]-\ln\left[4M_S\tau+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)\right]\right\}+\frac{3\tau^2}{2mx_B}\left(p_0-\frac{M_B}{x_B}\right)+\frac{3[Q_BL-(M_S-M_B)]}{2mL^2}\tau^2 \quad (54)$$

By the following equation

$$\dot{\theta}(t)=\alpha(t)=\frac{3\tau^3}{2m}\frac{\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^3}{\left[4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)\right]^2}$$

we can get the angle of AH:

$$\theta(t)=\frac{3\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^3\tau^3}{8mM_S}\left(\frac{1}{4M_S\tau+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)}-\frac{1}{4M_S\tau+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S\right)}\right)+\frac{3\tau^2}{2mx_B^2}\left(p_0-\frac{M_B}{x_B}\right)+\frac{3[Q_BL-(M_S-M_B)]}{2mL^3}\tau^2 \quad (55)$$

If $t_0=\left[\left(1-\frac{x_B}{2L}\right)+\frac{M_B}{4M_S}\left(3+4\frac{L}{x_B}+2\frac{x_B}{L}\right)+\frac{p_0L}{4M_S}\left(2-\frac{x_B}{L}\right)\right]\tau$, then

$$\theta_0=\frac{3}{8mM_S}\left(\frac{1}{x(\tau)}-\frac{1}{L+x_B}\right)\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^2\tau^2+\frac{3\tau^2}{2mx_B^2}\left(p_0-\frac{M_B}{x_B}\right)+\frac{3[Q_BL-(M_S-M_B)]}{2mL^3}\tau^2 \quad (56)$$

$$Z_0=\frac{3\left(p_0+\frac{M_B}{x_B}-2\frac{M_S-M_B}{L}\right)^2\tau^2}{8mM_S}\ln\frac{L+x_B}{x(\tau)}+\frac{3\tau^2}{2mx_B}\left(p_0-\frac{M_B}{x_B}\right)+\frac{3[Q_BL-(M_S-M_B)]}{2mL^2}\tau^2 \quad (57)$$

(c) For $t_0 \le t \le t_1$

Plastic hinge lies on the root of CB at this stage, so we get

$$\dot{z}(t)=\frac{2p_0\tau}{m(L+x_B)}-\frac{3p_S}{m(L+x_B)}(t-t_0)=\frac{3}{m(L+x_B)}\left[p_0\tau-p_St+\left(\frac{3M_B(L+x_B)}{2Lx_B}-\frac{3M_B}{2(L+x_B)}+\frac{p_S(2L-x_B)}{2L}-\frac{p_S(L+x_B)}{L}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right] \quad (58)$$

$$\alpha(t)=\frac{2p_0\tau}{m(L+x_B)^2}-\frac{3p_S}{m(L+x_B)^2}(t-t_0)=\frac{3}{m(L+x_B)^2}\left[p_0\tau-p_St+\left(\frac{3M_B(L+x_B)}{2Lx_B}-\frac{3M_B}{2(L+x_B)}+\frac{p_S(2L-x_B)}{2L}-\frac{p_S(L+x_B)}{L}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right] \quad (59)$$

If

$$\alpha(t)=\frac{3}{m(L+x_B)^2}\left[p_0\tau-p_St+\left(\frac{3M_B(L+x_B)}{2Lx_B}-\frac{3M_B}{2(L+x_B)}+\frac{p_S(2L-x_B)}{2L}-\frac{p_S(L+x_B)}{L}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right]=0$$

the CB stops motion at the moment $t_1=t_0+\frac{2p_0}{3p_S}\tau$.

For $t_0 \le t \le t_1=\left[\frac{p_0}{p_S}-\frac{3}{2}\frac{x_B}{L}-\frac{3}{2}\frac{M_B}{M_S}\cdot\frac{L^2+x_B^2+Lx_B}{L\cdot x_B}-\frac{3}{4}\frac{p_0x_B}{M_S}\right]\tau$, plastic hinge is stationary which stands on the root of CB.

Utilizing $\dot{z}(t)=\frac{3}{m(L+x_B)}\left[p_0\tau-p_St+\left(\frac{3M_B(L+x_B)}{2Lx_B}-\frac{3M_B}{2(L+x_B)}+\frac{p_S(2L-x_B)}{2L}-\frac{p_S(L+x_B)}{L}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right]$, we get following:

$$Z(t)=\frac{3t}{m(L+x_B)}\left[p_0\tau-\frac{1}{2}p_St+\left(\frac{3M_B(L+x_B)}{2Lx_B}-\frac{3M_B}{2(L+x_B)}+\frac{p_S(2L-x_B)}{2L}-\frac{p_S(L+x_B)}{L}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right]+Z_0 \quad (60)$$

$$\dot{\theta}(t)=\alpha(t)=\frac{3}{m(L+x_B)^2}\left[p_0\tau-p_St+\left(\frac{3M_B(L+x_B)}{2Lx_B}-\frac{3M_B}{2(L+x_B)}+\frac{p_S(2L-x_B)}{2L}-\frac{p_S(L+x_B)}{L}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right] \quad (61)$$

When $t=t_1=\left[\frac{p_0}{p_S}-\frac{3}{2}\frac{x_B}{L}-\frac{3}{2}\frac{M_B}{M_S}\cdot\frac{L^2+x_B^2+L\cdot x_B}{L\cdot x_B}-\frac{3}{4}\frac{p_0x_B}{M_S}\right]\tau$, then $\theta_1=\frac{3p_St^2}{2m(L+x_B)^2}+\theta_0$ and $Z(t)=\frac{3p_St_1^2}{2m(L+x_B)}+Z_0$.

3.3 $p_0=3\frac{M_B}{x_B}$

(a) For $0 \le t \le \tau$

At this stage, motion pattern of the CB is the same as the former cases. At the critical state, a plastic hinge appears only at the crack B on the CB. At the time $t=\tau$, compared with initial time, the momentum variation of the whole CB is:

$$\frac{1}{2}mx_B\cdot\frac{3t}{mx_B}\left(p_0-\frac{M_B}{x_B}\right)\Big|_{t=\tau}=p_0\tau \quad (62)$$

and the variation of angular momentum:

$$\frac{1}{6}mx_B^2\cdot\frac{3t}{mx_B}\left(p_0-\frac{M_B}{x_B}\right)\Big|_{t=\tau}=M_B\tau \quad (63)$$

In this phase there is only one stationary plastic hinge at its crack of the CB. The linear displacement of free end of CB is

$$Z(t)=\frac{3t^2}{2m\,x_B}\left(p_0-\frac{M_B}{x_B}\right) \quad (64)$$

and the angular displacement of segment AH

$$\theta(t)=\frac{3t^2}{2m\,x_B^2}\left(p_0-\frac{M_B}{x_B}\right) \quad (65)$$

(b) For $\tau \le t \le t_0$

In this case of no external loads, there are not singular plastic areas in CB and the CB is the single hinge mode like crack-free beam. However, for the sake of difference of boundary conditions, the performance of CB at this stage shows the unique features as follows.

As per the law of conservation of momentum and angular momentum for the CB, we get below

$$\begin{cases} \frac{1}{2}\mathrm{m}x_0(t)\dot{z}\Big|_{t=t} - \frac{1}{2}\mathrm{m}x_0(t)\dot{z}\Big|_{t=\tau} = \int_\tau^t P(t)dt \\ \frac{1}{6}\mathrm{m}x_0^2(t)\dot{z}\Big|_{t=t} - \frac{1}{6}\mathrm{m}x_0^2(t)\dot{z}\Big|_{t=\tau} = M_S(t-\tau) \end{cases} \tag{66}$$

and

$$\begin{cases} \frac{1}{2}\mathrm{m}x_0(t)\dot{z} = p_0\tau \\ \frac{1}{6}\mathrm{m}x_0^2(t)\dot{z} = M_S(t-\tau) + M_B\tau \end{cases} \tag{67}$$

According to Eq. (67), when $t=\tau$, then $x_0(\tau) = \frac{3M_B}{p_0} = x_B$, which means that the plastic hinge starts to migrate from position B at the moment of loads ending and no jumping phenomenon happens. At the time $t_0 = \left(\frac{p_0 L}{3M_S} + 1\right)\tau$, where t_0 corresponding to $x(t) = L + x_B$, we obtain the speed of free end A as below

$$\dot{z}(t_0) = \frac{2p_0\tau}{m\,(L+x_B)} \tag{68}$$

and the angular velocity of CB

$$\alpha(t_0) = \frac{2p_0\tau}{m\,(L+x_B)^2} \tag{69}$$

and the accelerated angular velocity

$$\omega(t_0) = -\frac{12M_S}{m\,(L+x_B)^3} \tag{70}$$

During this phase, when $x_0(t) = \frac{3M_S t - 3(M_S - M_B)\tau}{p_0\tau} = -\frac{3(M_S - M_B)}{p_0} + \frac{3M_S}{p_0}\cdot\frac{t}{\tau}$ there is a migrating plastic hinge with the constant migrating speed $v_H = \frac{dx(t)}{dt} = \frac{3M_S}{p_0}\cdot\frac{1}{\tau}$.

The linear displacement of free end of CB is

$$Z(t) = \frac{2p_0^2\tau^2}{3mM_S}\{\ln[M_S t - (M_S - M_B)\tau] - \ln[M_S\tau - (M_S - M_B)\tau]\} + \frac{3\tau^2}{2mx_B}\left(p_0 - \frac{M_B}{x_B}\right) \tag{71}$$

and the angular displacement of segment AH

$$\theta(t) = \frac{2p_0^3\tau^3}{9mM_S}\left(\frac{1}{M_B\tau} - \frac{1}{M_S t - (M_S - M_B)\tau}\right) + \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) \tag{72}$$

(c) For $t_0 \le t \le t_1$

During this phase, plastic hinge lies on the root, motion equations are showed as below

$$\dot{z}(t) = \frac{2p_0\tau}{m\,(L+x_B)} - \frac{3p_S}{m\,(L+x_B)}(t-t_0) = \frac{3}{m\,(L+x_B)}\left(p_0\tau - p_S t + \frac{M_S - M_B}{L+x_B}\tau\right) \tag{73}$$

$$\alpha(t) = \frac{2p_0\tau}{m\,(L+x_B)^2} - \frac{3p_S}{m\,(L+x_B)^2}(t-t_0) = \frac{3}{m\,(L+x_B)^2}\left(p_0\tau - p_S t + \frac{M_S - M_B}{L+x_B}\tau\right) \tag{74}$$

$$\omega(t) = -\frac{3p_S}{m\,(L+x_B)^2} \tag{75}$$

When $\alpha(t) = \frac{3}{m\,(L+x_B)^2}\left(p_0\tau - p_S t + \frac{M_S - M_B}{L+x_B}\tau\right) = 0$, then the motion of CB stops. During this phase, there is a stationary hinge on the root of CB.

The linear displacement of free end of CB is

$$Z(t) = \frac{3t}{m(L+x_B)}\left(p_0\tau - \frac{1}{2}p_S t + \frac{M_S - M_B}{L+x_B}\tau\right) + Z_0 \tag{76}$$

and the angular displacement of segment AO

$$\theta(t) = \frac{3t}{m(L+x_B)^2}\left(p_0\tau - \frac{1}{2}p_S t + \frac{M_S - M_B}{L+x_B}\tau\right) + \theta_0 \tag{77}$$

3.4 $p_0 > 3\frac{M_B}{x_B}$ and $3\frac{M_S+M_B}{Q_B} < L$

(a) For $0 \le t \le \tau$

At this phase, the motion pattern of CB under suddenly applied loads is the same as the former. For segment AB

$$\dot{Z}(t) = \frac{3t}{m\,x_B}\left(p_0 - \frac{M_B}{x_B}\right) \tag{78}$$

$$\alpha(t) = \frac{3t}{m\,x_B^2}\left(p_0 - \frac{M_B}{x_B}\right) \tag{79}$$

$$\omega(t) = \frac{3}{m\,x_B^2}\left(p_0 - \frac{M_B}{x_B}\right) \tag{80}$$

and for segment BH

$$x_0 = 3\frac{M_S+M_B}{Q_B} \tag{81}$$

$$\dot{Z}(t) = \frac{2Q_B^2 t}{3m(M_S+M_B)} \tag{82}$$

$$\alpha(t) = \frac{2Q_B^3 t}{9m(M_S+M_B)^2} \tag{83}$$

$$\omega(t) = \frac{2Q_B^3}{9m(M_S+M_B)^2} \tag{84}$$

When $t=\tau$, the variation of totally momentum of CB is

$$\frac{1}{2}mx_B \cdot \frac{3t}{m\,x_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{1}{2}m \cdot 3\frac{M_S+M_B}{Q_B} \cdot \frac{2Q_B^2 t}{3m(M_S+M_B)}\bigg|_{t=\tau} = p_0\tau \tag{85}$$

and the variation of angular momentum

$$\frac{1}{6}mx_B^2 \cdot \frac{3t}{m\,x_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{1}{6}m \cdot 9\frac{(M_S+M_B)^2}{Q_B^2} \cdot \frac{2Q_B^2 t}{3m(M_S+M_B)}\bigg|_{t=\tau} = \frac{\tau}{2}(p_0 x_B + M_B + 2M_S) \tag{86}$$

During this phase, there are two stationary plastic hinges on CB, one of which lies on its crack and another on the location $x_0 = 3\frac{M_S+M_B}{Q_B}$. The linear displacement of free end of CB can be expressed as:

$$Z(t) = \frac{3t^2}{2mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^2 t^2}{3m(M_S+M_B)} \tag{87}$$

and the angular displacement of segment AH

$$\theta(t) = \frac{3t^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3 t^2}{9m(M_S+M_B)^2} \tag{88}$$

(b) For $\tau \le t \le t_0$

At this phase, there is single hinge on the CB like with crack-free beam. However, for the sake of differences in boundary conditions, the CB shows the following unique features.

When $t=\tau$, $x_0(\tau) = \frac{1}{2}\left(x_B + \frac{2M_S+M_B}{p_0}\right)$ and $x_0(\tau) \ne x_0 = 3\frac{M_S+M_B}{Q_B}$ in general under the condition $x_0(\tau) \ne x_B$, which means that the jumping phenomenon of plastic hinge on the CB will happen at the moment of load end.

As per $x_0(t) = \frac{3}{2}\left(x_B + 3\frac{M_B}{p_0}\right) + \frac{3M_S}{p_0}\cdot\frac{t}{\tau}$, a migrating plastic hinge with constant speed $v_H = \frac{3M_S}{p_0}\cdot\frac{1}{\tau}$ occurs on the CB. We can also obtain linear displacement of CB as below

$$Z(t) = \frac{2p_0^2\tau^2}{3m}\left\{\ln\left[M_S t + \frac{\tau}{2}(p_0 x_B + 3M_B)\right] - \ln\left[M_S\tau + \frac{\tau}{2}(p_0 x_B + 3M_B)\right]\right\} + \frac{3\tau^2}{2mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^2\tau^2}{3m(M_S+M_B)} \tag{89}$$

and angular displacement of segment AH

$$\theta(t) = \frac{2p_0^3\tau^3}{9mM_S}\left(\frac{1}{M_S\tau+\frac{\tau}{2}(p_0x_B+3M_B)} - \frac{1}{M_S t+\frac{\tau}{2}(p_0x_B+3M_B)}\right) + \frac{3\tau^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{Q_B^3\tau^2}{9m(M_S+M_B)^2} \tag{90}$$

(c) For $t_0 \le t \le t_1$

During this phase, the motion pattern of CB is showed as following.

$$\dot{Z}(t) = \frac{3}{m(L+x_B)}\left(p_0\tau - p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right) \tag{91}$$

$$\alpha(t) = \frac{3}{m(L+x_B)^2}\left(p_0\tau - p_S t + \frac{3M_B - p_0 x_B}{2(L+x_B)}\tau\right) \tag{92}$$

$$\omega(t) = -\frac{3p_S}{m(L+x_B)^2} \quad (93)$$

According to equation (92), we know that the motion of CB stops at the moment $t = t_1 = t_0 + \frac{2p_0}{3p_S}\tau$.

3.5 $p_0 > 3\frac{M_B}{x_B}$ and $\frac{M_S+M_B}{Q_B} \le L \le 3\frac{M_S+M_B}{Q_B}$

(1) For $0 \le t \le \tau$

At this phase, the motion pattern of segment AB is depicted below,

$$\dot{Z}(t) = \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) \quad (94)$$

$$\alpha(t) = \frac{3t}{mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) \quad (95)$$

$$\omega(t) = \frac{3}{mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) \quad (96)$$

and for segment BO,

$$\dot{Z}(t) = \frac{3[Q_BL-(M_B+M_S)]}{mL^2}t \quad (97)$$

$$\alpha(t) = \frac{3[Q_BL-(M_B+M_S)]}{mL^3}t \quad (98)$$

$$\omega(t) = \frac{3[Q_BL-(M_B+M_S)]}{mL^3} \quad (99)$$

The variation of momentum of CB at the time $t = \tau$ is

$$\frac{1}{2}mx_B \cdot \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{1}{2}mL \cdot \frac{3[Q_BL-(M_S+M_B)]t}{mL^2}\bigg|_{t=\tau} = \frac{3\tau}{4}\left(p_0 + \frac{M_B}{x_B} - 2\frac{M_S+M_B}{L}\right) \quad (100)$$

and the variation of angular momentum

$$\frac{1}{6}mx_B^2 \cdot \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{1}{6}mL^2 \cdot \frac{3[Q_BL-(M_S+M_B)]t}{mL^2}\bigg|_{t=\tau} = \frac{\tau}{4}\left(3M_B\frac{L}{x_B} + 2p_0x_B - p_0L - 2M_S - 4M_B\right) \quad (101)$$

The linear velocity of free end of CB can be written as

$$\dot{Z}(t) = \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{3[Q_BL-(M_S+M_B)]}{mL^2}t \quad (102)$$

the angular displacement of segment AH

$$\theta(t) = \frac{3t^2}{2mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) + \frac{3[Q_BL-(M_S+M_B)]}{2mL^3}t^2 \quad (103)$$

(2) For $\tau \le t \le t_0$

During this phase, we can derive the motion pattern of CB as below:

$$x_0(t) = \frac{4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S-4M_B\right)}{\tau\left(p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}\right)} \quad (104)$$

$$\dot{Z}(t) = \frac{3\tau^2}{2m}\frac{\left(p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}\right)^2}{4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S-4M_B\right)} \quad (105)$$

$$\alpha(t) = \frac{3\tau^3}{2m}\frac{\left(p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}\right)^3}{\left[4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S-4M_B\right)\right]^2} \quad (106)$$

$$\omega(t) = \frac{12\tau^3}{m}\frac{\left(p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}\right)^3M_S}{\left[4M_St+\tau\left(3M_B\frac{L}{x_B}+2p_0x_B-p_0L-6M_S-4M_B\right)\right]^3} \quad (107)$$

From Eq. (104), it is easy to know that when $t = \tau$ then $x_0(\tau) = \frac{3M_B\frac{L}{x_B}+2p_0x_B-p_0L-2M_S-4M_B}{p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}}$. Since $x_0(\tau) \ne x_B$ and $x_0(\tau) \ne x_0 = L$ generally, a jumping phenomenon for the location of plastic hinge will happen at the moment of loading end. In formula mentioned above, t_0 means the moment of $x(t) = L + x_B$, then we can get

$$t_0 = \left[\left(1 - \frac{x_B}{2L}\right) + \frac{M_B}{4M_S}\left(3 + 4\frac{L}{x_B} - 2\frac{x_B}{L}\right) + \frac{p_0L}{4M_S}\left(2 - \frac{x_B}{L}\right)\right]\tau \quad (108)$$

Now the speed of free end A is

$$\dot{Z}(t_0)=\frac{3\left(p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}\right)\tau}{2m(L+x_B)} \tag{109}$$

and the angular velocity of CB

$$\alpha(t_0)=\frac{3\left(p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}\right)\tau}{2m(L+x_B)^2} \tag{110}$$

During this phase, there is one migrating plastic hinge on the CB and with constant migration speed $v_H=\frac{4M_S}{p_0+\frac{M_B}{x_B}-2\frac{M_S+M_B}{L}}\cdot\frac{1}{\tau}$.

(3) For $t_0\le t\le t_1$

There is one stationary plastic hinge on the root of CB during this phase. We can derive that

$$\dot{Z}(t)=\frac{3}{m(L+x_B)}\left[p_0\tau-p_St+\left(\frac{M_B(2L-3x_B)(L+2x_B)}{4(L+x_B)Lx_B}-\frac{3p_S(L^2+x_B^2)}{2L(L+x_B)}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right] \tag{111}$$

$$\alpha(t)\frac{3}{m(L+x_B)^2}\left[p_0\tau-p_St+\left(\frac{M_B(2L-3x_B)(L+2x_B)}{4(L+x_B)Lx_B}-\frac{3p_S(L^2+x_B^2)}{2L(L+x_B)}-\frac{3}{4}\cdot\frac{p_0x_B}{L+x_B}\right)\tau\right] \tag{112}$$

The linear displacement of CB is

$$Z(t)=\frac{3t}{m(L+x_B)}\left[p_0\tau-\frac{1}{2}p_St+\left(\frac{M_B(2L-3x_B)(L+2x_B)}{4(L+x_B)Lx_B}-\frac{3p_S(L^2+x_B^2)}{2L(L+x_B)}-\frac{3}{4}\frac{p_0x_B}{L+x_B}\right)\tau\right]+Z_0 \tag{113}$$

and the angular displacement of segment AO

$$\theta(t)=\frac{3t}{m(L+x_B)^2}\left[p_0\tau-\frac{1}{2}p_St+\left(\frac{M_B(2L-3x_B)(L+2x_B)}{4(L+x_B)Lx_B}-\frac{3p_S(L^2+x_B^2)}{2L(L+x_B)}-\frac{3}{4}\frac{p_0x_B}{L+x_B}\right)\tau\right]+\theta_0 \tag{114}$$

When $t=t_1=\frac{3p_St^2}{2m(L+x_B)^2}+\theta_0$, the motion of CB terminates.

4. Illustration of the Deformation Mode of CB Under Three Load Levels

A cantilever with 6000mm long and rectangular cross-section $200\times600mm^2$ is made of Q235 steel, as shown in Figure 4.1. There is a I-crack, whose length a is 100mm and 1000mm apart from the free end of CB, on the CB. The step impact loads, shown in Figure 4.2, are imposed on the free end of CB. Under the action of various impact loads size, the CB will present different deformation modes, as given by the above analysis. We will illustrate the deformation of CB under three states with different load magnitude as follows.

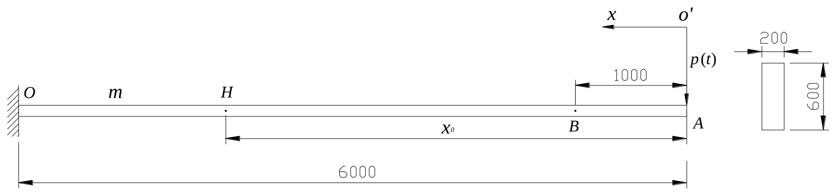
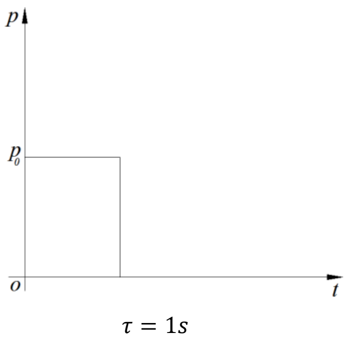


Figure 4.1 Schematic diagram of a CB



$\tau=1s$

Figure 4.2 Loading pattern

The main physical parameters of the beam material used in this example are given in Table 4.1.

Table 4.1 Physical parameters for a CB

Elastic modulus, E(Pa)	Bulk density, ρ(kg·m^{-3})	Yield strength σ_s(Pa)	Sectional momoent of inertia, I(m^4)	Ultimate moment, M_s(kN · m)	Ultimate moment at crack, M_B(kN · m)	Ultimate load, P_s(kN)
2.0×10^8	7800	2.35×10^{10}	7.2×10^{-3}	4230	3760	705

(1) When the impact load $p_0 = 13000KN$ and due to $p_0 > 3\frac{p_S}{x_B}L$, the crack lies in the plastic zone and its effect to the CB can be ignored. In this case, there is a stationary hinge on the CB and the plastic hinge moves toward the root of the CB during the process of loading. According to a different time scale, we can present their respective deformation modes of the CB as follows.

(a) When $0 \le t \le \tau$

$x(t) = \frac{3p_S}{p_0}L = \frac{3\times4230}{13000} = 0.97\text{m}$;

$\dot{Z}(t) = \frac{2p_0^2 t}{3mp_S L} = \frac{2\times13000^2 t}{3\times936\times4230} = 28.46t\ \text{m/s}$;

$\alpha(t) = \frac{2p_0^3 t}{9mp_S^2 L^2} = \frac{2\times13000^3 t}{9\times936\times4230^2} = 29.15t\ \text{1/s}$;

$\omega(t) = \frac{2p_0^3}{9mp_S^2 L^2} = \frac{2\times13000^3}{9\times936\times4230^2} = 29.15\ \text{1/s}^2$;

(b) When $\tau \le t \le t_0 = \frac{p_0}{3p_S}\tau = \frac{13000}{3\times705} = 6.15s$

$x(t) = \frac{3p_S}{p_0}\cdot\frac{t}{\tau}L = \frac{3\times705}{13000}\times\frac{t}{1}\times6 = 0.98t\ \text{m}$;

$\dot{Z}(t) = \frac{2p_0^2\tau^2}{3mp_S L\cdot t} = \frac{2\times13000^2}{3\times936\times705\times6\times t} = 28.46\frac{1}{t}\ \text{m/s}$;

$\alpha(t) = \frac{2p_0^3\tau^3}{9mp_S^2\cdot L^2\cdot t^2} = \frac{2\times13000^3}{9\times936\times705^2\times6^2\times t^2} = 19.15\ \frac{1}{t^2}\ \text{1/s}$;

$\omega(t) = -\frac{4p_0^3}{9mp_S^2 L^2}\left(\frac{\tau}{t}\right)^3 = \frac{4\times13000^3}{9\times936\times705^2\times6^2}\left(\frac{1}{t}\right)^3 = -58.30\frac{1}{t^3}\ \ \text{1/s}^2$;

(c) When $t_0 < t \le t_1 = \frac{p_0}{p_S}\tau = \frac{13000}{705} = 18.44s$

$x(t) = L = 6\text{m}$;

$\dot{Z}(t) = \frac{3}{mL}(p_0\tau - p_S t) = \frac{3}{936\times6}(13000\times1 - 705t) = 6.94 - 0.38t\ \text{m/s}$;

$\alpha(t) = \frac{3}{mL^2}(p_0\tau - p_S t) = \frac{3}{936\times6^2}(13000\times1 - 705t) = 1.16 - 0.06t\ \text{1/s}$;

$\omega(t) = -\frac{3p_S}{mL^2} = -\frac{3\times705}{936\times6^2} = -0.06\ \text{1/s}^2$;

(2) Under the condition of that the impact load is 5000kN and $3\frac{M_S - M_B}{Q_B} = 3\times\frac{4230-3760}{3140} < L = 5$, there are two stationary plastic hinges in the CB which one lies on the location of crack and another on root of CB during the period of loading. At the moment of loading end, the location of plastic hinge will change suddenly and only one moving hinge toward the root of CB occurs in the CB.

(a) When $0 \le t \le \tau$

For the segment AB of CB:

$\dot{Z}(t) = \frac{3t}{mx_B}\left(p_0 - \frac{M_B}{x_B}\right) = \frac{3t}{936}(5000 - 3760) = 3.97t\ \text{m/s}$;

$\alpha(t) = \frac{3t}{mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) = \frac{3t}{936}(5000 - 3760) = 3.97t^{\ 2}\ \ \text{1/s}$;

$\omega(t) = \frac{3}{mx_B^2}\left(p_0 - \frac{M_B}{x_B}\right) = \frac{3}{936}(5000 - 3760) = 3.97\ \ \text{1/s}$;

For the segment BH of CB:

$x_0 = 3\frac{M_S - M_B}{Q_B} = 3\times\frac{4230-3760}{3140} = 0.45m$;

$$\dot{Z}(t)=\frac{2Q_B^2 t}{3m(M_S-M_B)}=\frac{2\times3140^2 t}{3\times936\times(4230-3760)}=14.94t \quad m/s;$$

$$\alpha(t)=\frac{2Q_B^3 t}{9m(M_S-M_B)^2}=\frac{2\times3140^3 t}{9\times936\times(4230-3760)^2}=33.27t \quad 1/s;$$

$$\omega(t)=\frac{2Q_B^3}{9m(M_S-M_B)^2}=\frac{2\times3140^3}{9\times936\times(4230-3760)^2}=33.27;$$

(b) $\tau\le t\le t_0=\frac{p_0\tau}{3M_S}\left(L+\frac{9}{2}\frac{M_B}{p_0}-\frac{1}{2}x_B\right)=\frac{5000}{3\times4230}\left(5+\frac{9}{2}\times\frac{3760}{705}-\frac{1}{2}\right)=11.23\text{s}$

$$x_0(t)=\frac{3M_S t+\frac{3\tau}{2}(p_0x_B-3M_B)}{p_0\tau}=\frac{3\times4230t+\frac{3}{2}(5000-3\times3760)}{5000}=2.54t-1.88m;$$

$$\dot{Z}(t)=\frac{2p_0^2\tau^2}{3m\left[M_S t+\frac{\tau}{2}(p_0x_B-3M_B)\right]}=\frac{2\times5000^2}{3\times936\left[4230\times t+\frac{1}{2}(5000-3\times3760)\right]}=\frac{17806.27}{4230t-3140}m/s;$$

$$\alpha(t)=\frac{2p_0^3\tau^3}{9m\left[M_S t+\frac{\tau}{2}(p_0x_B-3M_B)\right]^2}=\frac{2\times5000^3}{9\times936\left[4230\times t+\frac{1}{2}(5000-3\times3760)\right]^2}=\frac{29677113.01}{(4230t-3140)^2}\quad 1/s;$$

$$\omega(t)=-\frac{4M_S p_0^3\tau^3}{9m\left[M_S t+\frac{\tau}{2}(p_0x_B-3M_B)\right]^3}=-\frac{4\times4230\times5000^3}{9\times936\left[4230\times t+\frac{1}{2}(5000-3\times3760)\right]^3}=-\frac{251068376100}{(4230t-3140)^3}\quad 1/s^2;$$

(c) $t_0<t\le t_1=t_0+\frac{2p_0}{3p_S}\tau=11.23+\frac{2\times5000}{3\times705}=15.96s$

$$\dot{Z}(t)=\frac{2p_0\tau}{m(L+x_B)}-\frac{3p_S(t-t_0)}{m(L+x_B)}=\frac{2\times5000}{936\times6}-\frac{3\times705\times(t-11.23)}{936\times6}=6.00-0.38t\ m/s;$$

$$\alpha(t)=\frac{2p_0\tau}{m(L+x_B)^2}-\frac{3p_S(t-t_0)}{m(L+x_B)^2}=\frac{2\times5000}{936\times6^2}-\frac{3\times705\times(t-11.23)}{936\times6^2}=1.00-0.06t\ \ 1/s;$$

$$\omega(t)=-\frac{3p_S}{m(L+x_B)^2}=-\frac{3\times705}{936\times6^2}=-0.06\ \ 1/s;$$

5. Conclusion and Discussion

5.1 Conclusion

This paper makes a elaborate study towards mechanical reaction of crack-beam under the action of step impact-load. Based on the above theoretical analysis, we conclude that the influence on the CB exercised by crack can be ignored for certain loading range and specific form and length of crack. This conclusion comes from the fact that the crack effect on the CB lies on the inherent nature of the crack and beam material: the yield stress σ_s, stress intensity factor K, the length of crack a and the height of beam H, but have no dependencies on other factors. If the length of crack is tiny, we can ignore the crack effect and the CB can be deal with as a crack-free beam. The influence of crack effect on the displacement of CB has two ways: the one is to significantly increase the magnitude of displacement of CB, especially for angular displacement and the movement duration is twice or more as long as crack-free beam, as showed in the example of this paper.

5.2 Discussions

5.2.1 Response Analysis Considering the Shear Effect

The effects of transverse shear on CB have been neglected in the above discussion, which might be rational in the analysis of statically plastic behavior of CB with small height-span ratio (HSR). However, when applied to those short beams with large HSR, the neglect of shear effect could cause large error. We have already had a relatively deep understanding to this kind of problem, while it's still not very clear about the error caused when analyzing the dynamically plastic behavior of small HSR beam without considering shear effect.

Likewise, we take the modified Parkes model, in which material properties is the same as before, and P is a suddenly imposed force. The function of shear force and moment is then

$$\begin{cases} Q(x)=p_0\left[1-\left(\frac{x}{x_0}\right)^2\right] \\ M(x)=p_0x\left[1-\frac{2}{3}\left(\frac{x}{x_0}\right)^2\right] \end{cases}$$

in which p_s is the ultimate loading, and $p_s L = M_s$ is the shear force needed to form a wholly plastic section. We have

$$\begin{cases} \frac{Q(x)}{Q_S} = \frac{p_0}{Q_S}\left[1-\left(\frac{x}{x_0}\right)^2\right] \\ \frac{M(x)}{M_S} = \frac{p_0 x}{p_s L}\left[1-\frac{2}{3}\left(\frac{x}{x_0}\right)^2\right] \quad 0 \le x \le x_0 \end{cases}$$

which should also satisfy following equations

$$\begin{cases} -1 \le \frac{Q(x)}{Q_S} \le 1 \\ -1 \le \frac{M(x)}{M_S} \le 1 \end{cases}$$

by above equations we can derive that:

When $p_0 \le Q_S$, then $-1 \le \frac{Q(x)}{Q_S} \le 1$, which means shear failure won't appear. When $p_0 > Q_S$, from the equation $\frac{p_0}{Q_S}\left[1-\left(\frac{x}{x_0}\right)^2\right] = 1$, we can get

$$x_1 = x_0\sqrt{1-\frac{Q_S}{p_0}} = \frac{3M_S}{p_0}\sqrt{1-\frac{Q_S}{p_0}} = \frac{3M_S}{p_0^{\frac{3}{2}}}\sqrt{p_0 - Q_S} \quad (p_0 > Q_S)$$

The condition, $-1 \le \frac{Q(x)}{Q_S} \le 1$, can't be satisfied at $x < x_1$,which implies that the beam will be broken as shear failure. When $p_0 \to \infty$, then $x_1 \to x_0$, thus shear failure will appear at the free end.

According to the above discussions, we can get the following conclusions:

1. With the participation of inertial forces force, the equation $\frac{\partial M(x)}{\partial x} = Q(x)$ will not hold in the infinitesimal beam segment. Referring to the equations of $Q(x)$ and $M(x)$, we get $\frac{\partial M(x)}{\partial x} = p_0\left[1-2\left(\frac{x}{x_0}\right)^2\right] < Q(x) = p_0\left[1-\left(\frac{x}{x_0}\right)^2\right]$, namely, when the inertial force gets involved, shear effect will increase and the rigid-plastic beam under dynamic loads will more likely to be broken as shear failure than that under static loads.

2. When $p_0 \le Q_S$, then $\left|\frac{Q(x)}{Q_S}\right| \le 1$, which means shear failure will never appear under whatever conditions so long as the external load is smaller than the ultimate shear force.

3. When $p_0 > Q_S$, then $x_1 = x_0\sqrt{1-\frac{Q_S}{p_0}} < x_0$,which means shear failure could only appear inside the segment AH, while inside the plastic zone (segment HO), it will never appear under whatever conditions are.

4. When $p_0 \to \infty$, then $x_1 = x_0 \to 0$, that is, when the sudden-applied force goes to infinity, only the free end tip would be broken.

Consider a critical condition $p_0 = Q_S$, that is $\frac{Q(x)}{Q_S} = \left[1-\left(\frac{x}{x_0}\right)^2\right]$. Under this circumstance, ultimate shear force can only be reached at the free end. The position of plastic hinge is at $x_0 = \frac{3M_S}{p_0} = \frac{3M_S}{Q_S}$, angular acceleration of segment AH is $\omega = \frac{2Q_S^3}{9mM_S^2}$. $\frac{M_S}{Q_S}$ is a constant to beams with the section of a specific material. For instance, as to a beam section $b \times h$, we have $M_S = \frac{1}{4}bh^2\sigma_0$ and $Q_S = \frac{2}{3}bh\tau_0$. According to Von Mises yield criterion, $\frac{\sigma_0}{\tau_0} = \sqrt{3}$ stands, and $\frac{M_S}{Q_S} = \frac{3\sqrt{3}}{8}H$, that is, the interface between plastic and rigid zone relates only to the height of the beam, while has nothing to do with the other factors.

Consider next the condition , assume that the shear slip hinge appears at point B, the length of segment AB is x_1, then according to the equilibrium equation, we have

$$\begin{cases} p - Q_S = m x_1 \dot{z}\frac{1}{2}(1+\eta) \\ p x_1 + M_B = \frac{1}{3} m x_1^2 (1-\eta)\dot{z} \end{cases}$$

For whole length of a beam, the moment is undetermined at the shear slip hinge ; While for segment AB, the only two equations we have gotten can't compute three unknown variables ,namely, the length of segment AB, velocity at the free end and point B. Therefore, we can't compute the above equations.

5.2.2 Response Analysis When the Crack Region is Considered as a Plastic Zone

What should be noted is that there will be a large development of plastic zone in the region of certain cracks under possible dynamic loads. When the plastic zone develops to the degree that can't be neglected comparing to the length of the beam, a certain error will be caused by the assumption of the localization and centralization of crack effects. Although we have tried to probe into this problem, confined by the limitations of mathematical and mechanical treatment, it’s hard to make available headway.

In the above analysis, the determinant of crack region considering plastic redistribution is based on the assumption of infinite body, when applied to finite body problems, the difference of boundary conditions can be treated with an approximate stress intensity factor if the interface of plastic and rigid regions is inside the body (Approximate solution is also adopted in practical applications since the accurate stress intensity factor of finite body problems is still unobtainable). While it is assumed to be of one dimension in the analysis of beam-like members, so a segment of the beam is assumed to be in plastic region only if the whole section of this segment entered into plasticity. The reason of neglecting crack effects in this paper is also based on this consensus. However, when the crack effects are not negligible, that is, when the interface between plastic and rigid regions is found out to be intersecting with the geometrical boundary according to the infinite-body assumption, the plastic redistribution caused by geometric configuration will have to be considered. Besides, how to choose the right criterion for this kind of plastic redistribution is also a tough problem.

Even a reliable criterion can be chosen, based on which we can gain a new function of the boundary line between plastic and rigid regions, a judgment according to this criterion to decide whether the whole section of a critical region of certain cracks has entered into plasticity should also be obtained. However, because of the complexity of boundary line function, there are some difficulties in mathematical treatment.

The assumption of localization and centralization of crack effects in this paper is tenable in most cases, because the influence of the scale of plastic regions, which is obtained as the above discussion, to the whole-beam deformation mode is almost equivalent to the plastic hinge at a crack point.

5.2.3 Large Geometric Deformation Effect

The second order effects such as strain ratio, shear force, strain hardening, large geometric deformation and etc. were neglected in the analysis. The above discussion on shear effects has disclosed that a certain degree of error can be caused when the second order effects are ignored. The following discussion will be on the influence of large geometric deformations.

It can be seen from the above example that the small-deformation assumption caused a great error in computation, which has reached an order of the beam length. Three points should be made here especially:

1. The gap between the actual displacement of a particle and that from small-deformation assumption: As illustrated in the graph, because the involving of inertial force aside with a large rotation angle, this gap will have a great impact on the result.

2. The influence of axial force effect: Since this paper chose a cantilever, which is a statically determinate structure, there is no “membrane effect” or “arch effect” brought by the axial force after deformation, which means that this kind of influence doesn’t exist in this paper. However, in the common used beams with two ends hinged or fixed, or even in the rigid framework structure, the effect of axial force can always bring a substantial impact, especially in the condition of large deformations, which can even become the main effect transcending that of the moment sometimes.

3. Computational outcomes of the above example has shown that since a large impact load was chosen and the duration of load was also relatively large, the resulting line displacements and rotation

displacements are all considerable. Because we didn't do limit analysis in this paper, this result may only work as a reference for the quantitative comparison between the responses of different models.

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