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Bending and Torsional Buckling Analysis of C-shaped Columns based on Plate-Beam Theory

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Abstract

Among various cross-sections, the C-shaped section has been increasingly applied in construction projects due to its advantages of smooth and beautiful shape, good integrity, and convenient connection. However, theoretical research on the buckling problem of C-shaped components is limited. The plate-beam theory combined with the Euler beam model and Kirchhof plate model to establish an energy variational model. This article derives the bending and torsional buckling model of C-shaped cross-sectional members within the framework of plate beam theory. The results show that the bending and torsional buckling of C-shaped axially compressed members are decoupled, and the control equation does not include the cross coupling term. This theoretical result is of great significance for the design and application of C-section components.

Keywords

C-shaped Section Member; Bending and Torsional Buckling; Plate-beam Theory; Decoupled.

1. Introduction

From the early 20th century to the present, over the past 100 years, a large amount of effective basic and applied research has been conducted on the elastic bending torsion buckling theory of open thin-walled components. Representative buckling theories include the traditional theory of zero mid plane shear strain represented by Vlasov and Bleich, the theory of zero total shear strain in the middle plane, such as nonlinear shear strain energy proposed by Lv Liewu and Shen Shizhao, Pi&Trahair theory, theory of zero total shear strain in the middle plane represented by Ghobarah&Tso, transverse normal stress theory proposed Hugue&Ma and Tong Genshu. The achievements of these theories are the basis for the current formulation of standards and the stability design of engineering structures. However, Research has found that these theories have the following issues: Firstly, the derivation of all theories follows the concept of Vlasov sector coordinate or Wagner unit warping. The application of Vlasov sector coordinate is the difficulty in the thin-walled components, since Vlasov sector coordinate belongs to the unconventional coordinate system, difficult to understand. Secondly, The displacement expressions of any point on the cross-section is established based on the basic assumptions, zero mid plane shear strain, not well referencing existing engineering mechanics achievements, such as mature thin plate theory and beam theory. Thirdly, according to the displacement expression, all strain energy cannot be naturally derived, such as free torsional strain

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energy and Secondary warping strain energy, which need to be manually supplement during the derivation process, otherwise it will lead to incorrect conclusions.

As well known, Kirchhof thin plate theory and Euler beam theory are the most widely used and mature engineering mechanics theories in engineering, with the advantages of high accuracy and easy understanding. The plate-beam theory is a new engineering theory proposed by Professor Zhang Wenfu[1-7], which can solve problems such as combined torsion and bending torsion buckling of thin-walled components. The theory contains three basic assumptions: hypothesis of rigid cross-section, hypothesis of plate deformation and hypothesis of beam deformation. Among them, hypothesis of rigid cross-section is the basic one first proposed by Vlasov to describe the deformation theory of thin-walled members. The plate deformation assumption and beam deformation assumption are first proposed based on the principle of plate deformation decomposition. The out-of-plane deformation of the member is described by Kirchhoff-plate theory, while the in-plane deformation is described by Euler-beam theory. Different from the classical Vlasov theory, the longitudinal displacement, strain and strain energy can be derived from the mature Kirchhoff-plate theory and Euler-beam theory, which can solve the problem of representing the warping function generated during the composite of multiple materials.

The derivation process of the plate-beam theory is rational and thus avoids being arbitrary and any resulting unnecessary controversy. Moreover, theoretical practice [8-10] shows that plate-beam theory can deal with steel-concrete composite plates, open/closed cross-sections, and combined cross-sections. In this paper, a new theory of open thin-wall member with C-type cross-section elastic bending is established based on plate-beam theory.

2. Basic Hypotheses

2.1 Hypothesis of Rigid Cross-section

Hypothesis of rigid cross-section means that the contour of the crosssection is undeformable, i.e. the cross-section is rigid after the occurrence of the beam lateral buckling.

This is the well-known peripheral rigidity hypothesis of Vlasov, which means that the local and distortional buckling are excluded in the LTB model. It also shows that the shear deformation of a plate in the tangential plane (i.e. the middle surface) caused by the in-plane bending is ignored.

2.2 Hypothesis of Neglecting Shear Deformation

Hypothesis of neglecting shear deformation means that, the deformation of the in-plane bending and that of the out-of-plane bending and torsion of each flat plate (either a web or flange plate) can be described by the Euler-beam model and Kirchhoff-plate model, respectively.

3. Energy Equation Derivation via Plate-beam Theory

3.1 Coordinate Systems and Geometric Quantities

For loss of generality, the C-shaped aluminum section alloy column shown in Figure 1 is the research object.

In order to describe the deformation, the plate-beam theory needs to introduce two sets of coordinate systems: overall coordinate system xyz and local coordinate system nsz. These two sets of coordinate systems are similar to those of Vlasov theory. Both sets of coordinate systems follow the right-handed spiral rule. The origin of the overall coordinate system is selected in the center of the section, and the origin of the local coordinate system of each plate is selected in the center of the plate.

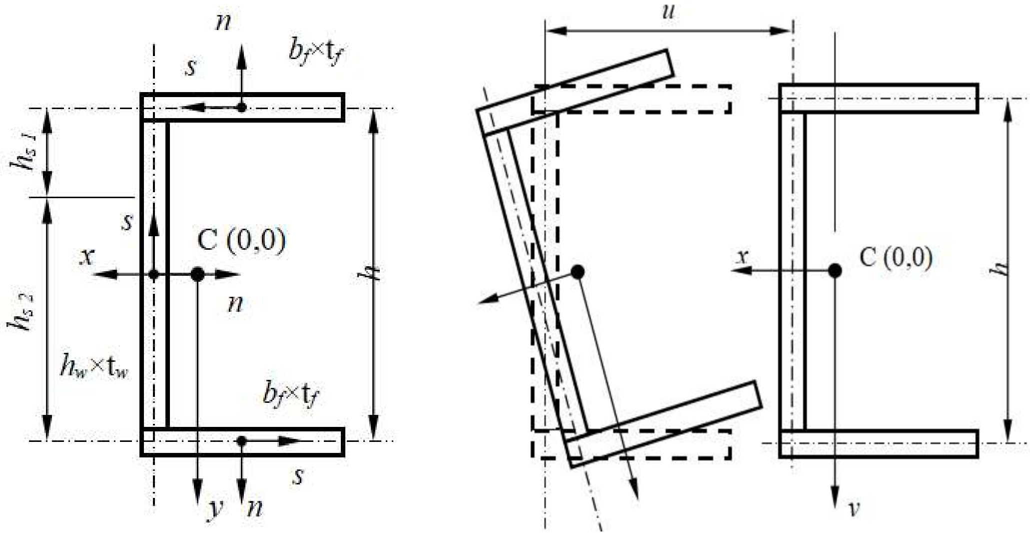

Figure 1. Coordinate systems and deformation diagram of C-type section

It is known that the length of the C-shaped section column is L ; the height and thickness of the web are h_w and t_w, respectively; the breadth and thickness of the flange are b_f and t_f, respectively; the elastic modulus is E ; shear modulus is G ; and Poisson's ratio is μ .

If the C-shaped column bends under the action of axial pressure, there are two unknown amounts of the cross section, the corner around the shear core $\theta(z)$ and the side of the shear core (center) moving along the axis x, $u(z)$.

Length h is a geometric quantity that we often use in the beam and column buckling theory for the distance between the upper and lower flanges. Note the definition of the total height of the section $h = h_w + t_f$, the distance between the flange and the web itself to the center of the section $h = h_w + 2t_f$, and the coordinate of the shear core (e_f, e_w). It should be noted that the distance between the center and the shear heart y_0, and the diagram is negative under the overall coordinate system.

3.2 Strain energy of bending and torsion buckling

3.2.1 Strain Energy of the Web

(1) Deformation of the web

The center coordinate of the combined section is C(0,0). The overall coordinate of the web center is B$(e_w, 0)$, where the origin of the web local coordinate is located at the web center. The displacement of any point under the local coordinate system (n, s) is:

$$\alpha = \frac{3\pi}{2},\ x - x_0 = -n - e,\ y - y_0 = -s \tag{1}$$

Where, α is the corner from the axis n to the overall coordinate system y . The coordinate transformation relationship is as:

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$$\begin{pmatrix} r_s \\ r_n \end{pmatrix} = \begin{pmatrix} \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} \\ \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} \end{pmatrix} \begin{pmatrix} -n-e \\ -s \end{pmatrix} = \begin{pmatrix} n+e \\ s \end{pmatrix} \tag{2}$$

Displacement conversion relationship is:

$$\begin{pmatrix} v_s \\ v_n \\ \theta \end{pmatrix} = \begin{pmatrix} \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} & r_s \\ \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} & -r_n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ \theta \end{pmatrix} = \begin{pmatrix} (n+e)\theta \\ -u-s\theta \\ \theta \end{pmatrix} \tag{3}$$

The displacement of the web center is:

$$u_{\mathrm{w}}^0 = -u,\ v_{\mathrm{w}}^0 = e\theta,\ w_{\mathrm{w}}^0 = 0,\ \theta_{\mathrm{w}}^0 = 0 \tag{4}$$

In particular, u_{w}^0, v_{w}^0, w_{w}^0 are the web center displacements on directions of the axis n, axis s and axis z of the local coordinate system.

(2) Displacement field of the web

a. In-plane bending (no in-plane bending).

b. Out-of-plane bending (Kirchhoff-plate model).

According to the principle of deformation decomposition, the displacement pattern outside the web plane is:

Displacement along the axis n.

$$u_{\mathrm{w}}(s,z) = -u + s\theta \tag{5}$$

Displacement along the axis s.

$$v_{\mathrm{w}}(n,z) = (n+e)\theta \tag{6}$$

The longitudinal displacement (the displacement along the axis z) needs to be determined according to the Kirchhoff-plate model, i. E.

$$\begin{aligned} w_{\mathrm{w}}(n,s,z) &= w_{\mathrm{w}}^0 - n\left(\frac{\partial u_{\mathrm{w}}}{\partial z}\right) \\ &= -n\left[-\frac{\partial u}{\partial z} - s\left(\frac{\partial \theta}{\partial z}\right)\right] = n\left(\frac{\partial u}{\partial z}\right) + ns\left(\frac{\partial \theta}{\partial z}\right) \end{aligned} \tag{7}$$

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(3) Geometric equation of the web (linear strain)

$$\varepsilon_{z,\mathrm{w}} = \frac{\partial w_{\mathrm{w}}}{\partial z} = n\left(\frac{\partial^2 u}{\partial z^2}\right) + ns\left(\frac{\partial^2 \theta}{\partial z^2}\right) \tag{8}$$

$$\varepsilon_{s,\mathrm{w}} = \frac{\partial v_{\mathrm{w}}}{\partial s} = 0 \tag{9}$$

$$\gamma_{sz,\mathrm{w}} = \frac{\partial w_{\mathrm{w}}}{\partial s} + \frac{\partial v_{\mathrm{w}}}{\partial z} = n\left(\frac{\partial \theta}{\partial z}\right) + (n+e)\left(\frac{\partial \theta}{\partial z}\right) = (2n+e)\left(\frac{\partial \theta}{\partial z}\right) \tag{10}$$

(4) Physical equations

For the Kirchhoff-plate model, there is:

$$\sigma_{z,\mathrm{w}} = \frac{E}{1-\mu^2}\left(\varepsilon_{z,\mathrm{w}}\right), \quad \tau_{sz,\mathrm{w}} = G\gamma_{sz,\mathrm{w}} \tag{11}$$

(5) The strain energy of the web

a. Strain energy of the in-plane bending

$$U_{\mathrm{w}}^{\text{in-plane}} = 0 \tag{12}$$

b. Strain energy for out-of-plane bending (Kirchhoff-plate model)

On the basis of:

$$U = \frac{1}{2}\iiint\left[\frac{E}{1-\mu^2}\left(\varepsilon_{z,\mathrm{w}}^2\right) + G\gamma_{sz,\mathrm{w}}^2\right]\mathrm{d}n\mathrm{d}s\mathrm{d}z,$$

We have:

$$U_{\mathrm{w}}^{\text{out-plane}} = \frac{1}{2}\iiint\left\{\frac{E}{1-\mu^2}\left[n\left(\frac{\partial^2 u}{\partial z^2}\right) + ns\left(\frac{\partial^2 \theta}{\partial z^2}\right)\right]^2 + G\left[(2n+e)\left(\frac{\partial \theta}{\partial z}\right)\right]^2\right\}\mathrm{d}n\mathrm{d}s\mathrm{d}z \tag{13}$$

In which,

$$\iint_{A_{\mathrm{w}}} n^2 \mathrm{d}n\mathrm{d}s = \int_{-\frac{h_{\mathrm{w}}}{2}}^{\frac{h_{\mathrm{w}}}{2}}\int_{-\frac{t_{\mathrm{w}}}{2}}^{\frac{t_{\mathrm{w}}}{2}} n^2 \mathrm{d}n\mathrm{d}s = \frac{h_{\mathrm{w}} t_{\mathrm{w}}^3}{12}$$

$$\iint_{A_{\mathrm{w}}} n^2 s^2 \mathrm{d}n\mathrm{d}s = \int_{-\frac{h_{\mathrm{w}}}{2}}^{\frac{h_{\mathrm{w}}}{2}}\int_{-\frac{t_{\mathrm{w}}}{2}}^{\frac{t_{\mathrm{w}}}{2}} n^2 s^2 \mathrm{d}n\mathrm{d}s = \frac{h_{\mathrm{w}}^3 t_{\mathrm{w}}^3}{144}$$

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The available integral result is as:

$$U_{\mathrm{w}}^{\text{out-plane}}=\frac{1}{2}\int_0^L\left\{\frac{E}{1-\mu^2}\left[\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{12}\right)\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\left(\frac{h_{\mathrm{w}}^3t_{\mathrm{w}}^3}{144}\right)\left(\frac{\partial^2\theta}{\partial z^2}\right)^2\right]+G\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{3}+e^2A_{\mathrm{w}}\right)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}\mathrm{d}z \tag{14}$$

For slender beams and strips, because of μ^2 far less than 1. Therefore, in the derivation of Vlasov, using the simplified relationship,

$$U_{\mathrm{w}}^{\text{out-plane}}=\frac{1}{2}\int_0^L\left[\left(EI_y\right)_{\mathrm{w}}\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\left(EI_{\mathrm{w}}\right)_{\mathrm{w}}\left(\frac{\partial^2\theta}{\partial z^2}\right)^2+\left(GJ_{\mathrm{k}}\right)_{\mathrm{w}}\left(\frac{\partial\theta}{\partial z}\right)^2\right]\mathrm{d}z \tag{15}$$

In which:

$$\left(EI_y\right)_{\mathrm{w}}=E\left(\frac{t_{\mathrm{w}}^3h_{\mathrm{w}}}{12}\right)$$

$$\left(EI_{\mathrm{w}}\right)_{\mathrm{w}}=E\left(\frac{h_{\mathrm{w}}^3t_{\mathrm{w}}^3}{144}\right)$$

$$\left(GJ_{\mathrm{k}}\right)_{\mathrm{w}}=G\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{3}+e^2A_{\mathrm{w}}\right)$$

3.2.2 Strain Energy of the Upper Flange

(1) The deformation of the upper flange

The overall coordinate of the center of the upper flange is $\left(x_{cf},-\frac{h}{2}\right)$, where $x_{\mathrm{cf}}=-\left(\frac{b_{\mathrm{f}}}{2}-\frac{t_{\mathrm{w}}}{2}-e_{\mathrm{w}}\right)$.

The displacement of any point in the local coordinate (n,s) of the upper flange is:

$$\alpha=2\pi,\ x-x_0=s+x_{\mathrm{cf}},\ y-y_0=-\frac{h}{2}-n \tag{16}$$

$$\begin{pmatrix}r_s\\ r_n\end{pmatrix}=\begin{pmatrix}\sin 2\pi & -\cos 2\pi\\ \cos 2\pi & \sin 2\pi\end{pmatrix}\begin{pmatrix}s+x_{\mathrm{cf}}\\ -n-\frac{h}{2}\end{pmatrix}=\begin{pmatrix}n+\frac{h}{2}\\ s+x_{\mathrm{cf}}\end{pmatrix} \tag{17}$$

$$\begin{pmatrix}v_s\\ v_n\\ \theta\end{pmatrix}=\begin{pmatrix}\cos 0 & \sin 0 & r_s\\ \sin 0 & -\cos 0 & -r_n\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}u\\ 0\\ \theta\end{pmatrix}=\begin{pmatrix}u+\theta\left(n+\frac{h}{2}\right)\\ -\theta\left(s+x_{\mathrm{cf}}\right)\\ \theta\end{pmatrix} \tag{18}$$

The displacement of the center of the upper flange is:

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$$u_{\mathrm{f}}^{0}=0,\ v_{\mathrm{f}}^{0}=u+\frac{h}{2}\theta,\ w_{\mathrm{f}}^{0}=0,\ \theta_{\mathrm{f0}}^{0}=\theta \tag{19}$$

In which, u_{f}^{0}, v_{f}^{0}, and w_{f}^{0} are the flange center displaced along the axis n, axis s and axis z of the local coordinate system.
According to the principle of deformation decomposition, the variable form (15.223) can be decomposed into:

$$\begin{pmatrix} v_s \\ v_n \\ \theta \end{pmatrix}=\begin{pmatrix} u+\dfrac{h}{2}\theta \\ 0 \\ 0 \end{pmatrix}_{\text{in-plane}}+\begin{pmatrix} n\theta \\ -(s+x_{\mathrm{cf}})\theta \\ \theta \end{pmatrix}_{\text{out-plane}} \tag{20}$$

(2) In-plane strain energy of bending of the upper flange (Euler-beam model)
According to the principle of deformation decomposition, the displacement pattern in the upper flange is:
Displacement along the axis n:

$$u_{\mathrm{f}}(z)=u_{\mathrm{f}}^{0}=0 \tag{21}$$

Displacement along the axis s:

$$v_{\mathrm{f}}(z)=v_{\mathrm{f}}^{0}=u+\frac{h}{2}\theta \tag{22}$$

The longitudinal displacement (the displacement along the axis z) needs to be determined by the Euler-beam model, i. E:

$$\begin{aligned} w_{\mathrm{f}}(s,z)&=u_{\mathrm{f}}^{0}-\frac{\partial v_{\mathrm{f}}^{0}}{\partial z}s \\ &=-\frac{\partial}{\partial z}\left[u+\frac{h}{2}\theta\right]s=-s\left[\frac{\partial u}{\partial z}+\frac{h}{2}\frac{\partial\theta}{\partial z}\right] \end{aligned} \tag{23}$$

Geometric equation (Linear strain):

$$\varepsilon_{z,\mathrm{f1}}=\frac{\partial w_{\mathrm{f}}}{\partial z}=-s\left[\frac{\partial^2 u}{\partial z^2}+\frac{h}{2}\frac{\partial^2\theta}{\partial z^2}\right],\ \varepsilon_{s,\mathrm{f}}=\frac{\partial v_{\mathrm{f}}}{\partial s}=0 \tag{24}$$

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$$\gamma_{sz,\mathrm{f}}=\frac{\partial w_{\mathrm{f}}}{\partial s}+\frac{\partial v_{\mathrm{f}}}{\partial z}=-\left[\frac{\partial u}{\partial z}+\frac{h}{2}\frac{\partial \theta}{\partial z}\right]+\left[\frac{\partial u}{\partial z}+\frac{h}{2}\frac{\partial \theta}{\partial z}\right]=0 \tag{25}$$

Physical equation:

$$\left.\begin{aligned}\sigma_{z,\mathrm{f}}&=E\varepsilon_{z,\mathrm{f}}\\ \tau_{sz,\mathrm{f}}&=G\gamma_{sz,\mathrm{f}}\end{aligned}\right\} \tag{26}$$

Strain energy:

$$U=\frac{1}{2}\iiint\left(\sigma_z\varepsilon_z+\sigma_s\varepsilon_s+\tau_{sz}\gamma_{sz}\right)\mathrm{d}n\mathrm{d}s\mathrm{d}z=\frac{1}{2}\iiint\left(\sigma_{z,\mathrm{f}}\varepsilon_{z,\mathrm{f}}\right)\mathrm{d}n\mathrm{d}s\mathrm{d}z \tag{27}$$

Then,

$$\begin{aligned}U_{\mathrm{f,top}}^{\text{in-plane}}&=\frac{1}{2}\iiint_{V_{\mathrm{f1}}}E\left\{-s\left[\frac{\partial^2 u}{\partial z^2}+\left(\frac{h}{2}\right)\frac{\partial^2\theta}{\partial z^2}\right]\right\}^2\mathrm{d}n\mathrm{d}s\mathrm{d}z\\ &=4\times\frac{1}{2}\int_0^L\int_0^{\frac{t_{\mathrm{f}}}{2}}\int_0^{\frac{b_{\mathrm{f}}}{2}}E\left\{-s\left[\frac{\partial^2 u}{\partial z^2}+\left(\frac{h}{2}\right)\frac{\partial^2\theta}{\partial z^2}\right]\right\}^2\mathrm{d}s\mathrm{d}n\mathrm{d}z\end{aligned} \tag{28}$$

The integral result is:

$$U_{\mathrm{f,top}}^{\text{in-plane}}=\frac{1}{2}\int_0^L\left[\left(EI_y\right)_{\mathrm{f1}}\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\left(EI_y\right)_{\mathrm{f1}}\frac{h^2}{4}\left(\frac{\partial^2\theta}{\partial z^2}\right)^2+\left(EI_y\right)_{\mathrm{f1}}h\left(\frac{\partial^2 u}{\partial z^2}\right)\left(\frac{\partial^2\theta}{\partial z^2}\right)^2\right]\mathrm{d}z \tag{29}$$

(3) Out-of -plane strain energy of bending of upper flange (Kirchhoff-plate model)
According to the principle of deformation decomposition, the displacement pattern of the outside of the plane of the upper flange is:
Displacement along the axis s:

$$u_{\mathrm{f}}(s,z)=-(s+x_{\mathrm{cf}})\theta \tag{30}$$

Displacement along the axis n:

$$v_{\mathrm{f}}(n,z)=n\theta \tag{31}$$

The longitudinal displacement (the displacement along the axis z) needs to be determined according to the Kirchhoff sh model, i. e:

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$$w_{\mathrm{f}}(n,s,z)=-n\left(\frac{\partial u_{\mathrm{f}}}{\partial z}\right)=n(s+x_{\mathrm{cf}})\left(\frac{\partial \theta}{\partial z}\right) \tag{32}$$

Geometric equations (Linear strain):

$$\varepsilon_{z,\mathrm{f}}=\frac{\partial w_{\mathrm{f}}}{\partial z}=n(s+x_{\mathrm{cf}})\left(\frac{\partial^2 \theta}{\partial z^2}\right),\ \ \varepsilon_{s,\mathrm{f}}=\frac{\partial v_{\mathrm{f}}}{\partial s}=0 \tag{33}$$

$$\gamma_{sz,\mathrm{f}}=\frac{\partial w_{\mathrm{f}}}{\partial s}+\frac{\partial v_{\mathrm{f}}}{\partial z}=n\left(\frac{\partial \theta}{\partial z}\right)+n\left(\frac{\partial \theta}{\partial z}\right)=2n\left(\frac{\partial \theta}{\partial z}\right) \tag{34}$$

Physical equations:

$$\left.\begin{aligned}\sigma_{z,\mathrm{f}}&=\frac{E}{1-\mu^2}\left(\varepsilon_{z,\mathrm{f}}\right)\\ \tau_{sz,\mathrm{f}}&=G\gamma_{sz,\mathrm{f}}\end{aligned}\right\} \tag{35}$$

Among them, $\frac{E}{1-\mu^2}$ is named subtracted elastic modulus of the length direction in Vlasov theory.

For slender beams and strips, the constitutive equation reduces to $\frac{E}{1-\mu^2}\approx E$, so that, strain energy of the upper flange caused by out-of-plane bending deformation and torsion,

$$U=\frac{1}{2}\iiint\left(\sigma_z\varepsilon_z+\sigma_s\varepsilon_s+\tau_{sz}\gamma_{sz}\right)\mathrm{d}n\mathrm{d}s\mathrm{d}z=\frac{1}{2}\iiint\left(\sigma_{z,\mathrm{f}}\varepsilon_{z,\mathrm{f}}+\gamma_{sz,\mathrm{f}}\tau_{sz,\mathrm{f}}\right)\mathrm{d}n\mathrm{d}s\mathrm{d}z \tag{36}$$

$$\begin{aligned}U_{\mathrm{f,top}}^{\text{out-plane}}&=\frac{1}{2}\iiint\left[E\left(\varepsilon_{z,\mathrm{f}}^2\right)+G\left(\gamma_{sz,\mathrm{f}}^2\right)\right]\mathrm{d}n\mathrm{d}s\mathrm{d}z\\ &=\frac{1}{2}\iiint\left\{E\left[n(s+x_{\mathrm{cf}})\left(\frac{\partial^2\theta}{\partial z^2}\right)\right]^2+G\left[2n\left(\frac{\partial\theta}{\partial z}\right)\right]^2\right\}\mathrm{d}n\mathrm{d}s\mathrm{d}z\\ &=\frac{1}{2}\iiint\left\{E\left[n^2s^2+2(n^2s)x_{\mathrm{cf}}+x_{\mathrm{cf}}{}^2\right]\left(\frac{\partial^2\theta}{\partial z^2}\right)^2+G\left(4n^2\right)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}\mathrm{d}n\mathrm{d}s\mathrm{d}z\end{aligned} \tag{37}$$

The integral result is:

$$U_{\mathrm{f,top}}^{\text{out-plane}}=\frac{1}{2}\int_0^L\left[\left(EI_{\mathrm{w}}\right)_{\mathrm{f1}}\left(\frac{\partial^2\theta}{\partial z^2}\right)^2+\left(EA_{\mathrm{f}}x_{\mathrm{cf}}^2\right)\left(\frac{\partial^2\theta}{\partial z^2}\right)^2+\left(GJ_{\mathrm{k}}\right)_{\mathrm{f1}}\left(\frac{\partial\theta}{\partial z}\right)^2\right]\mathrm{d}z \tag{38}$$

In which,

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$$\left(EI_{\mathrm{w}}\right)_{\mathrm{f1}}=E\left(\frac{t_{\mathrm{f}}^{3}b_{\mathrm{f}}^{3}}{144}\right)$$

is the second warping stiffness of the upper flange;

$$\left(GJ_{\mathrm{k}}\right)_{\mathrm{f1}}=G\left(\frac{b_{\mathrm{f}}t_{\mathrm{f}}^{3}}{3}\right)$$

is the St-Venant torsional stiffness of the upper flange.

3.2.3 Strain Energy of the Lower Flange

According to the calculation process of strain energy at the upper flange, the strain energy at the lower flange is obtained as:

$$U_{\mathrm{f,bottom}}^{\text{in-plane}}=\frac{1}{2}\int_{0}^{L}\left[\left(EI_{y}\right)_{\mathrm{f2}}\left(\frac{\partial^{2}u}{\partial z^{2}}\right)^{2}+\left(EI_{y}\right)_{\mathrm{f2}}\frac{h^{2}}{4}\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}-\left(EI_{y}\right)_{\mathrm{f2}}h\left(\frac{\partial^{2}u}{\partial z^{2}}\right)\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}\right]\mathrm{d}z \tag{39}$$

$$U_{\mathrm{f,bottom}}^{\text{out-plane}}=\frac{1}{2}\int_{0}^{L}\left[\left(EI_{\mathrm{w}}\right)_{\mathrm{f2}}\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}+\left(GJ_{\mathrm{k}}\right)_{\mathrm{f2}}\left(\frac{\partial\theta}{\partial z}\right)^{2}\right]\mathrm{d}z \tag{40}$$

3.3 Total Strain Energy of the Uniaxial Symmetric C-shaped Beam

$$U=U_{\mathrm{w}}+U_{\mathrm{f}}=U_{\mathrm{w}}^{\text{in-plane}}+U_{\mathrm{w}}^{\text{out-plane}}+U_{\mathrm{f1}}^{\text{in-plane}}+U_{\mathrm{f1}}^{\text{out-plane}}+U_{\mathrm{f2}}^{\text{in-plane}}+U_{\mathrm{f2}}^{\text{out-plane}} \tag{41}$$

In which,

$$U_{\mathrm{w}}^{\text{in-plane}}=0;$$

$$U_{\mathrm{w}}^{\text{out-plane}}=\frac{1}{2}\int_{0}^{L}\left[\left(EI_{y}\right)_{\mathrm{w}}\left(\frac{\partial^{2}u}{\partial z^{2}}\right)^{2}+\left(EI_{\mathrm{w}}\right)_{\mathrm{w}}\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}+\left(GJ_{\mathrm{k}}\right)_{\mathrm{w}}\left(\frac{\partial\theta}{\partial z}\right)^{2}\right]\mathrm{d}z;$$

$$U_{\mathrm{f,top}}^{\text{in-plane}}=\frac{1}{2}\int_{0}^{L}\left[\left(EI_{y}\right)_{\mathrm{f1}}\left(\frac{\partial^{2}u}{\partial z^{2}}\right)^{2}+\left(EI_{y}\right)_{\mathrm{f1}}\frac{h^{2}}{4}\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}+\left(EI_{y}\right)_{\mathrm{f1}}h\left(\frac{\partial^{2}u}{\partial z^{2}}\right)\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}\right]\mathrm{d}z;$$

$$U_{\mathrm{f,top}}^{\text{out-plane}}=\frac{1}{2}\int_{0}^{L}\left[\left(EI_{\mathrm{w}}\right)_{\mathrm{f1}}\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}+\left(EA_{\mathrm{f}}x_{\mathrm{cf}}^{2}\right)\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}+\left(GJ_{\mathrm{k}}\right)_{\mathrm{f1}}\left(\frac{\partial\theta}{\partial z}\right)^{2}\right]\mathrm{d}z;$$

$$U_{\mathrm{f,bottom}}^{\text{in-plane}}=\frac{1}{2}\int_{0}^{L}\left[\left(EI_{y}\right)_{\mathrm{f2}}\left(\frac{\partial^{2}u}{\partial z^{2}}\right)^{2}+\left(EI_{y}\right)_{\mathrm{f2}}\frac{h^{2}}{4}\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}-\left(EI_{y}\right)_{\mathrm{f2}}h\left(\frac{\partial^{2}u}{\partial z^{2}}\right)\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}\right]\mathrm{d}z;$$

$$U_{\mathrm{f,bottom}}^{\text{out-plane}}=\frac{1}{2}\int_{0}^{L}\left[\left(EI_{\mathrm{w}}\right)_{\mathrm{f2}}\left(\frac{\partial^{2}\theta}{\partial z^{2}}\right)^{2}+\left(GJ_{\mathrm{k}}\right)_{\mathrm{f2}}\left(\frac{\partial\theta}{\partial z}\right)^{2}\right]\mathrm{d}z;$$

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Then,

$$U=\frac{1}{2}\int_0^L\left\{\begin{array}{l}\left[\frac{E_{\mathrm{w}}}{1-\mu_{\mathrm{w}}^2}\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{12}\right)+\frac{E_{\mathrm{f}}}{1-\mu_{\mathrm{f}}^2}\left(\frac{t_{\mathrm{f}}b_{\mathrm{f}}^3}{12}\right)\right]\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\\ \left[\begin{array}{l}\frac{E_{\mathrm{w}}}{1-\mu_{\mathrm{w}}^2}\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{12}\right)(e_{\mathrm{w}}-y_0)^2+\frac{E_{\mathrm{f}}}{1-\mu_{\mathrm{f}}^2}(e_{\mathrm{f}}+y_0)^2\left(\frac{t_{\mathrm{f}}b_{\mathrm{f}}^3}{12}\right)+\\ \frac{E_{\mathrm{w}}}{1-\mu_{\mathrm{w}}^2}\left(\frac{h_{\mathrm{w}}^3t_{\mathrm{w}}^3}{144}\right)+\frac{E_{\mathrm{f}}}{1-\mu_{\mathrm{f}}^2}\left(\frac{t_{\mathrm{f}}^3b_{\mathrm{f}}^3}{144}\right)\end{array}\right]\left(\frac{\partial^2\theta}{\partial z^2}\right)^2+\\ \left[G_{\mathrm{w}}\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{3}\right)+G_{\mathrm{f}}\left(\frac{b_{\mathrm{f}}t_{\mathrm{f}}^3}{3}\right)\right]\left(\frac{\partial\theta}{\partial z}\right)^2+\\ \left[\frac{E_{\mathrm{w}}}{1-\mu_{\mathrm{w}}^2}2\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{12}\right)(-e_{\mathrm{w}}+y_0)+\frac{E_{\mathrm{f}}}{1-\mu_{\mathrm{f}}^2}2(e_{\mathrm{f}}+y_0)\left(\frac{t_{\mathrm{f}}b_{\mathrm{f}}^3}{12}\right)\right]\left(\frac{\partial^2 u}{\partial z^2}\right)\left(\frac{\partial^2\theta}{\partial z^2}\right)\end{array}\right\}\mathrm{d}z \tag{42}$$

Looking at this expression, we find that the last term $\left(\frac{\partial^2 u}{\partial z^2}\right)\left(\frac{\partial^2\theta}{\partial z^2}\right)$ of the strain energy is the influence of the cross term, which actually reflects the coupling effect of the bending and torsion deformations. However, if the unknown quantity $u(z)$ is selected, it is defined as the lateral shift of a special point (i. e. shear core) along the x axis, then the purpose of decoupling can be achieved. To formally eliminate the influence of this cross term, the last item in the above equation is zero,

$$\frac{E_{\mathrm{w}}}{1-\mu_{\mathrm{w}}^2}2\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{12}\right)(-e_{\mathrm{w}}+y_0)+\frac{E_{\mathrm{f}}}{1-\mu_{\mathrm{f}}^2}2(e_{\mathrm{f}}+y_0)\left(\frac{t_{\mathrm{f}}b_{\mathrm{f}}^3}{12}\right)=0 \tag{43}$$

Using geometric relations, the relationship between e_{w} and e_{f} is:

$$e_{\mathrm{w}}=\frac{h_{\mathrm{w}}+t_{\mathrm{f}}}{2}-e_{\mathrm{f}}\,(15.246) \tag{44}$$

Then, it takes:

$$h_{s1}=e_{\mathrm{f}}+y_0=\frac{\frac{E_{\mathrm{w}}}{1-\mu_{\mathrm{w}}^2}\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{12}\right)\left(\frac{h_{\mathrm{w}}+t_{\mathrm{f}}}{2}\right)}{\frac{E_{\mathrm{w}}}{1-\mu_{\mathrm{w}}^2}\left(\frac{h_{\mathrm{w}}t_{\mathrm{w}}^3}{12}\right)+\frac{E_{\mathrm{f}}}{1-\mu_{\mathrm{f}}^2}\left(\frac{t_{\mathrm{f}}b_{\mathrm{f}}^3}{12}\right)} \tag{45}$$

If take:

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$$\left(EI_y\right)_{\text{comp}} = \frac{E_{\text{w}}}{1-\mu_{\text{w}}^2}\left(\frac{h_{\text{w}} t_{\text{w}}^3}{12}\right) + \frac{E_{\text{f}}}{1-\mu_{\text{f}}^2}\left(\frac{t_{\text{f}} b_{\text{f}}^3}{12}\right)$$

as the flexural stiffness around the weak axis, while:

$$\left(EI_{\text{w}}\right)_{\text{comp}} = \frac{E_{\text{w}}}{1-\mu_{\text{w}}^2}\left(\frac{h_{\text{w}} t_{\text{w}}^3}{12}\right)\left(\frac{h_{\text{w}}+t_{\text{f}}}{2} - h_{s1}\right)^2 + \frac{E_{\text{f}}}{1-\mu_{\text{f}}^2}\left(h_{s1}\right)^2\left(\frac{t_{\text{f}} b_{\text{f}}^3}{12}\right) + \frac{E_{\text{w}}}{1-\mu_{\text{w}}^2}\left(\frac{h_{\text{w}}^3 t_{\text{w}}^3}{144}\right) + \frac{E_{\text{f}}}{1-\mu_{\text{f}}^2}\left(\frac{b_{\text{f}}^3 t_{\text{f}}^3}{144}\right) \tag{46}$$

For the constrained torsional stiffness or called warping stiffness, while:

$$\left(GJ_k\right)_{\text{comp}} = G_{\text{w}}\left(\frac{h_{\text{w}} t_{\text{w}}^3}{3}\right) + G_{\text{f}}\left(\frac{b_{\text{f}} t_{\text{f}}^3}{3}\right) \tag{47}$$

For free torsional stiffness, the total strain energy of a single-axial symmetric C-type column can be expressed as:

$$U = \frac{1}{2}\int_0^L \left[\left(EI_y\right)_{\text{comp}}\left(\frac{\partial^2 u}{\partial z^2}\right)^2 + \left(EI_y\right)_{\text{comp}}\left(\frac{\partial^2 \theta}{\partial z^2}\right)^2 + \left(GJ_{\text{k}}\right)_{\text{comp}}\left(\frac{\partial \theta}{\partial z}\right)^2\right]\text{d}z \tag{48}$$

4. Initial Stress Potential Energy of Torsion Flexion

4.1 Initial Stress and Potential Energy of the Web

Under axial pressure, the initial stress of the web is:

$$\sigma_{z,\text{w}0} = -\frac{E_{\text{w}} P}{\left(EA\right)_{\text{com}}} \tag{49}$$

In the formula,

$$\left(EA\right)_{\text{com}} = E_{\text{w}} A_{\text{w}} + E_{\text{f}} A_{\text{f}} \tag{50}$$

The positive and negative signs of stress are the same as those in the mechanics of materials, that is, the tensile stress is positive, while the compressive stress is negative.
The initial stress potential energy of the web is:

$$V_{\text{w}} = \iiint \left(\sigma_{z,\text{w}0}\,\varepsilon_{z,\text{w}}^{\text{NL}}\right)\text{d}n\,\text{d}s\,\text{d}z \tag{51}$$

In which,

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$$\varepsilon_{z,\mathrm{w}}^{\mathrm{NL}}=\frac{1}{2}\left[\left(\frac{\partial u_{\mathrm{w}}}{\partial z}\right)^2+\left(\frac{\partial v_{\mathrm{w}}}{\partial z}\right)^2\right]=\frac{1}{2}\left\{\left[-\left(\frac{\partial u}{\partial z}\right)+\left(-s+e_{\mathrm{w}}-y_0\right)\left(\frac{\partial \theta}{\partial z}\right)\right]^2+\left[n\left(\frac{\partial \theta}{\partial z}\right)\right]^2\right\} \tag{52}$$

So there is:

$$V_{\mathrm{w}}^{\text{out-plane}}=-\frac{E_{\mathrm{w}}P}{(EA)_{\mathrm{com}}}\iiint\frac{1}{2}\left\{\left[-\left(\frac{\partial u}{\partial z}\right)+\left(-s+e_{\mathrm{w}}-y_0\right)\left(\frac{\partial \theta}{\partial z}\right)\right]^2+\left[n\left(\frac{\partial \theta}{\partial z}\right)\right]^2\right\}\mathrm{d}n\mathrm{d}s\mathrm{d}z \tag{53}$$

The integral result is:

$$V_{\mathrm{w}}=-\frac{E_{\mathrm{w}}P}{(EA)_{\mathrm{com}}}\int_0^L\left[\begin{array}{l}\frac{1}{2}A_{\mathrm{w}}\left(\frac{\partial u}{\partial z}\right)^2-A_{\mathrm{w}}\Delta_1\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial \theta}{\partial z}\right)+\\ \left(\frac{1}{24}h_{\mathrm{w}}^3t_{\mathrm{w}}+\frac{1}{24}h_{\mathrm{w}}t_{\mathrm{w}}^3+\frac{1}{24}h_{\mathrm{w}}t_{\mathrm{w}}\Delta_1^2\right)\left(\frac{\partial \theta}{\partial z}\right)^2\end{array}\right]\mathrm{d}z \tag{54}$$

Where,

$$\Delta_1=e_{\mathrm{w}}-y_0$$

Cut the distance from the web itself to the section.

4.2 Initial Stress Potential Energy of the Upper Flange

Under the axial pressure state, the initial stress of the upper flange is:

$$\sigma_{z,\mathrm{w}0}=-\frac{E_{\mathrm{f}}P}{(EA)_{\mathrm{com}}} \tag{55}$$

In the formula,

$$(EA)_{\mathrm{com}}=E_{\mathrm{w}}A_{\mathrm{w}}+E_{\mathrm{f}}A_{\mathrm{f}} \tag{56}$$

According to the above provisions, the compressive stress of the upper flange shall be negative. The initial stress potential energy of the upper flange is:

$$V_{\mathrm{w}}=\iiint\left(\sigma_{z,\mathrm{f}0}\varepsilon_{z,\mathrm{f}}^{\mathrm{NL}}\right)\mathrm{d}n\mathrm{d}s\mathrm{d}z \tag{57}$$

Where:

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$$\varepsilon_{z,\mathrm{f}}^{\mathrm{NL}}=\frac{1}{2}\left[\left(\frac{\partial u_{\mathrm{f}}}{\partial z}\right)^{2}+\left(\frac{\partial v_{\mathrm{f}}}{\partial z}\right)^{2}\right]=\frac{1}{2}\left\{\left[\left(\frac{\partial u}{\partial z}\right)+\left(n+e_{\mathrm{f}}+y_{0}\right)\left(\frac{\partial\theta}{\partial z}\right)\right]^{2}+\left[s\left(\frac{\partial\theta}{\partial z}\right)\right]^{2}\right\} \tag{58}$$

So there is:

$$V_{\mathrm{f,top}}^{\text{in-plane}}=-\frac{E_{\mathrm{f}}P}{(EA)_{\mathrm{com}}}\iiint\frac{1}{2}\left\{\left[\left(\frac{\partial u}{\partial z}\right)+\left(n+e_{\mathrm{f}}+y_{0}\right)\left(\frac{\partial\theta}{\partial z}\right)\right]^{2}+\left[s\left(\frac{\partial\theta}{\partial z}\right)\right]^{2}\right\}\mathrm{d}n\mathrm{d}s\mathrm{d}z \tag{59}$$

The integral result is:

$$V_{\mathrm{f,top}}^{\text{in-plane}}=-\frac{E_{\mathrm{w}}P}{(EA)_{\mathrm{com}}}\int_{0}^{L}\left[\begin{array}{l}\frac{1}{2}A_{\mathrm{f}}\left(\frac{\partial u}{\partial z}\right)^{2}-A_{\mathrm{f}}\Delta_{1}\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right)+\\ \left(\frac{1}{24}b_{\mathrm{f}}^{3}t_{\mathrm{f}}+\frac{1}{24}b_{\mathrm{f}}t_{\mathrm{f}}^{3}+\frac{1}{24}b_{\mathrm{f}}t_{\mathrm{f}}\Delta_{2}^{2}\right)\left(\frac{\partial\theta}{\partial z}\right)^{2}\end{array}\right]\mathrm{d}z \tag{60}$$

Where,

$$\Delta_{2}=e_{\mathrm{f}}+y_{0}=h_{s1}$$

Is the distance from the center of the upper flange itself to the section.

4.3 Total Initial Stress Potential Energy

$$V=V_{\mathrm{w}}+V_{\mathrm{f}}$$

$$=\int_{0}^{L}\left\{\begin{array}{l}-\frac{E_{\mathrm{w}}P}{(EA)_{\mathrm{comp}}}\left[\begin{array}{l}\frac{1}{2}A_{\mathrm{w}}\left(\frac{\partial u}{\partial z}\right)^{2}-A_{\mathrm{w}}\Delta_{1}\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right)+\\ \left(\frac{1}{24}h_{\mathrm{w}}^{3}t_{\mathrm{w}}+\frac{1}{24}h_{\mathrm{w}}t_{\mathrm{w}}^{3}+\frac{1}{24}h_{\mathrm{w}}t_{\mathrm{w}}\Delta_{1}^{2}\right)\left(\frac{\partial\theta}{\partial z}\right)^{2}\end{array}\right]-\\ \frac{E_{\mathrm{w}}P}{(EA)_{\mathrm{comp}}}\left[\begin{array}{l}\frac{1}{2}A_{\mathrm{f}}\left(\frac{\partial u}{\partial z}\right)^{2}-A_{\mathrm{f}}\Delta_{1}\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right)+\\ \left(\frac{1}{24}b_{\mathrm{f}}^{3}t_{\mathrm{f}}+\frac{1}{24}b_{\mathrm{f}}t_{\mathrm{f}}^{3}+\frac{1}{24}b_{\mathrm{f}}t_{\mathrm{f}}\Delta_{2}^{2}\right)\left(\frac{\partial\theta}{\partial z}\right)^{2}\end{array}\right]\end{array}\right\}\mathrm{d}z \tag{61}$$

Perhaps:

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$$V=-\frac{1}{2}\int_0^L\left\{\begin{array}{l}\left[\frac{E_{\mathrm{w}}A_{\mathrm{w}}P}{(EA)_{\mathrm{comp}}}+\frac{E_{\mathrm{f}}A_{\mathrm{f}}P}{(EA)_{\mathrm{comp}}}\right]\left(\frac{\partial u}{\partial z}\right)^2+\\ \left[\begin{array}{l}\frac{E_{\mathrm{w}}P}{(EA)_{\mathrm{comp}}}\left(\frac{1}{12}h_{\mathrm{w}}^3t_{\mathrm{w}}+\frac{1}{12}h_{\mathrm{w}}t_{\mathrm{w}}^3+h_{\mathrm{w}}t_{\mathrm{w}}\varDelta_1^2\right)+\\ \frac{E_{\mathrm{f}}P}{(EA)_{\mathrm{comp}}}\left(\frac{1}{12}b_{\mathrm{f}}^3t_{\mathrm{f}}+\frac{1}{12}b_{\mathrm{f}}t_{\mathrm{f}}^3+b_{\mathrm{f}}t_{\mathrm{f}}\varDelta_2^2\right)\end{array}\right]\left(\frac{\partial\theta}{\partial z}\right)^2+\\ 2\left[-\frac{E_{\mathrm{w}}A_{\mathrm{w}}P}{(EA)_{\mathrm{comp}}}\varDelta_1+\frac{E_{\mathrm{f}}A_{\mathrm{f}}P}{(EA)_{\mathrm{comp}}}\varDelta_2\right]\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right)\end{array}\right\}\mathrm{d}z \tag{62}$$

The first term in the above equation can be reduced to:

$$\left[\frac{E_{\mathrm{w}}A_{\mathrm{w}}}{(EA)_{\mathrm{comp}}}+\frac{E_{\mathrm{f}}A_{\mathrm{f}}}{(EA)_{\mathrm{comp}}}\right]P\left(\frac{\partial u}{\partial z}\right)^2=P\left(\frac{\partial u}{\partial z}\right)^2 \tag{63}$$

Note the following relationship:

$$\begin{aligned}&E_{\mathrm{w}}h_{\mathrm{w}}t_{\mathrm{w}}\varDelta_1^2+E_{\mathrm{f}}b_{\mathrm{f}}t_{\mathrm{f}}\varDelta_2^2\\&=E_{\mathrm{w}}A_{\mathrm{w}}(e_{\mathrm{w}}-y_0)^2+E_{\mathrm{f}}A_{\mathrm{f}}(e_{\mathrm{f}}+y_0)^2\\&=\left(E_{\mathrm{w}}A_{\mathrm{w}}e_{\mathrm{w}}{}^2+E_{\mathrm{f}}A_{\mathrm{f}}e_{\mathrm{f}}{}^2\right)+\left(E_{\mathrm{w}}A_{\mathrm{w}}+E_{\mathrm{f}}A_{\mathrm{f}}\right)y_0{}^2+\left(E_{\mathrm{f}}A_{\mathrm{f}}e_{\mathrm{f}}-E_{\mathrm{w}}A_{\mathrm{w}}e_{\mathrm{w}}\right)\end{aligned} \tag{64}$$

From the condition of zero bending moment, it is known:

$$E_{\mathrm{f}}A_{\mathrm{f}}e_{\mathrm{f}}-E_{\mathrm{w}}A_{\mathrm{w}}e_{\mathrm{w}}=0 \tag{65}$$

Thus, formula (64) can be simplified to:

$$E_{\mathrm{w}}h_{\mathrm{w}}t_{\mathrm{w}}\varDelta_1^2+E_{\mathrm{f}}b_{\mathrm{f}}t_{\mathrm{f}}\varDelta_2^2=\left(E_{\mathrm{w}}A_{\mathrm{w}}e_{\mathrm{w}}{}^2+E_{\mathrm{f}}A_{\mathrm{f}}e_{\mathrm{f}}{}^2\right)+(EA)_{\mathrm{comp}}y_0{}^2 \tag{66}$$

where,

$$(EA)_{\mathrm{comp}}=E_{\mathrm{w}}A_{\mathrm{w}}+E_{\mathrm{f}}A_{\mathrm{f}} \tag{67}$$

According to equation (65), the second item can be simplified to:

$$\begin{bmatrix} \frac{E_{\mathrm{w}}P}{(EA)_{\mathrm{comp}}}\left(\frac{1}{12}h_{\mathrm{w}}^{3}t_{\mathrm{w}}+\frac{1}{12}h_{\mathrm{w}}t_{\mathrm{w}}^{3}+h_{\mathrm{w}}t_{\mathrm{w}}\Delta_{1}^{2}\right)+ \\ \frac{E_{\mathrm{f}}P}{(EA)_{\mathrm{comp}}}\left(\frac{1}{12}b_{\mathrm{f}}^{3}t_{\mathrm{f}}+\frac{1}{12}b_{\mathrm{f}}t_{\mathrm{f}}^{3}+b_{\mathrm{f}}t_{\mathrm{f}}\Delta_{2}^{2}\right) \end{bmatrix}\left(\frac{\partial\theta}{\partial z}\right)^{2}$$
$$=\frac{(EI_{x})_{\mathrm{comp}}+(EI_{y})_{\mathrm{comp}}+(EI_{y})_{\mathrm{comp}}y_{0}^{2}}{(EA)_{\mathrm{comp}}}P\left(\frac{\partial\theta}{\partial z}\right)^{2}=P(r_{p})_{\mathrm{comp}}^{2}\left(\frac{\partial\theta}{\partial z}\right)^{2} \tag{68}$$

$$\left.\begin{aligned} (EI_{x})_{\mathrm{comp}}&=E_{\mathrm{w}}\left(\frac{1}{12}h_{\mathrm{w}}^{3}t_{\mathrm{w}}\right)+E_{\mathrm{f}}\left(\frac{1}{12}b_{\mathrm{f}}t_{\mathrm{f}}^{3}\right)+\left(E_{\mathrm{w}}A_{\mathrm{w}}e_{\mathrm{w}}^{2}+E_{\mathrm{f}}A_{\mathrm{f}}e_{\mathrm{f}}^{2}\right) \\ (EI_{y})_{\mathrm{comp}}&=E_{\mathrm{w}}\left(\frac{1}{12}h_{\mathrm{w}}t_{\mathrm{w}}^{3}\right)+E_{\mathrm{f}}\left(\frac{1}{12}b_{\mathrm{f}}^{3}t_{\mathrm{f}}\right) \end{aligned}\right\} \tag{69}$$
$$(r_{p})_{\mathrm{comp}}^{2}=\frac{(EI_{x})_{\mathrm{comp}}+(EI_{y})_{\mathrm{comp}}+(EA)_{\mathrm{comp}}y_{0}^{2}}{(EA)_{\mathrm{comp}}}=\frac{(EI_{x})_{\mathrm{comp}}+(EI_{y})_{\mathrm{comp}}}{(EA)_{\mathrm{comp}}}+y_{0}^{2}$$

This is the square of the radius of the plate-beam theory. Unlike the traditional formula, this result can consider the case where the flange and web are different materials. In addition, all physical quantities, such as equation (66), are consistent with the exact solution of the mechanics of materials, while the Vlasov theory can only give an approximate formula.

The third item can be simplified to:

$$\left\{-\frac{E_{\mathrm{w}}A_{\mathrm{w}}}{(EA)_{\mathrm{comp}}}P\left[\frac{h_{\mathrm{w}}+t_{\mathrm{f1}}}{2}-(e_{\mathrm{f}}+y_{0})\right]+\frac{E_{\mathrm{f}}A_{\mathrm{f}}}{(EA)_{\mathrm{comp}}}P(e_{\mathrm{f}}+y_{0})\right\}\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right)$$
$$=\left\{-\frac{E_{\mathrm{w}}A_{\mathrm{w}}}{(EA)_{\mathrm{comp}}}P\left(\frac{h_{\mathrm{w}}+t_{\mathrm{f1}}}{2}\right)+\left[\frac{E_{\mathrm{w}}A_{\mathrm{w}}}{(EA)_{\mathrm{comp}}}+\frac{E_{\mathrm{f}}A_{\mathrm{f}}}{(EA)_{\mathrm{comp}}}\right]P(e_{\mathrm{f}}+y_{0})\right\}\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right) \tag{70}$$
$$=\left[-\frac{E_{\mathrm{w}}A_{\mathrm{w}}}{(EA)_{\mathrm{comp}}}P\left(\frac{h_{\mathrm{w}}+t_{\mathrm{f1}}}{2}\right)+P(e_{\mathrm{f}}+y_{0})\right]\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right)$$

Using the new definition of the centroid, i. e:

$$e_{\mathrm{f}}=\frac{E_{\mathrm{w}}A_{\mathrm{w}}\left(\frac{h_{w}+t_{\mathrm{f}}}{2}\right)}{(EA)_{\mathrm{comp}}} \tag{71}$$

The result of the third item is. $Py_{0}\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial\theta}{\partial z}\right)$.

To sum up, the initial stress potential energy can be written more simply as:

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$$V=-\frac{1}{2}\int_0^L\left[P\left(\frac{\partial u}{\partial z}\right)^2+P(r_p)^2_{\text{comp}}\left(\frac{\partial \theta}{\partial z}\right)^2+2Py_0\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial \theta}{\partial z}\right)\right]\mathrm{d}z \tag{72}$$

5. Energy Variation Model and Differential Equation Model

5.1 Energy Variation Model

The total potential energy of bending bending of T-shaped steel column is:

$$\Pi=U+V=\frac{1}{2}\int_0^L\left[\begin{array}{l}(EI_y)_{\text{comp}}\left(\frac{\partial^2 u}{\partial z^2}\right)^2+(EI_{\text{w}})_{\text{comp}}\left(\frac{\partial^2 \theta}{\partial z^2}\right)^2+(GJ_{\text{k}})_{\text{comp}}\left(\frac{\partial \theta}{\partial z}\right)^2-\\ P\left(\frac{\partial u}{\partial z}\right)^2-P(r_p)^2_{\text{comp}}\left(\frac{\partial \theta}{\partial z}\right)^2-2Py_0\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial \theta}{\partial z}\right)\end{array}\right]\mathrm{d}z \tag{73}$$

At this point, it can transform the twist-bending problem of C-section column into an energy variation model: find the sum of two functions $u(z)$ and $\theta(z)$ in the interval $0\le z\le L$, so that they meet the prescribed geometric boundary conditions, and make the following formula:

$$\Pi=\int_0^a F(u',u'',\theta',\theta'')\mathrm{d}x \tag{74}$$

Where:

$$F(u',u'',\theta',\theta'')=\frac{1}{2}\left[\begin{array}{l}(EI_y)_{\text{comp}}\left(\frac{\partial^2 u}{\partial z^2}\right)^2+(EI_{\text{w}})_{\text{comp}}\left(\frac{\partial^2 \theta}{\partial z^2}\right)^2+(GJ_{\text{k}})_{\text{comp}}\left(\frac{\partial \theta}{\partial z}\right)^2-\\ P\left(\frac{\partial u}{\partial z}\right)^2-P(r_p)^2_{\text{comp}}\left(\frac{\partial \theta}{\partial z}\right)^2-2Py_0\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial \theta}{\partial z}\right)\end{array}\right] \tag{75}$$

The defined energy functional takes the minimum value.

5.2 Differential Equation Model

According to the energy variation model derived above, the differential equation model of C-section can be easily introduced.

From all the derivatives of the functional $F(u',u'',\theta',\theta'')$:

$$F_u=\frac{\partial F}{\partial u}=0\,,\quad F_{u'}=\frac{\partial F}{\partial u'}=-Pu'-Py_0\theta'\,,\quad F_{u''}=\frac{\partial F}{\partial u''}=(EI_y)_{\text{comp}}u'' \tag{76}$$

$$F_\theta=\frac{\partial F}{\partial \theta}=0\,,\quad F_{\theta'}=\frac{\partial F}{\partial \theta'}=(GJ_{\text{k}})_{\text{comp}}\theta'-P(r_p)^2_{\text{comp}}\,Py_0u' \tag{77}$$

$$F_{\theta''}=\frac{\partial F}{\partial \theta''}=(EI_{\text{w}})_{\text{comp}}\theta'' \tag{78}$$

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The following Euler equations and boundary conditions are available.
Euler equation (equilibrium equation):

$$\left.\begin{aligned} F_u - \frac{\mathrm{d}}{\mathrm{d}z}F_{u'} + \frac{\mathrm{d}^2}{\mathrm{d}z^2}F_{u''} = 0 \\ F_\theta - \frac{\mathrm{d}}{\mathrm{d}z}F_{\theta'} + \frac{\mathrm{d}^2}{\mathrm{d}z^2}F_{\theta''} = 0 \end{aligned}\right\} \tag{79}$$

Perhaps:

$$\left.\begin{aligned} &(EI_y)_{\text{comp}}u^{(4)} + Pu'' + Py_0\theta'' = 0 \\ &(EI_{\text{w}})_{\text{comp}}\theta^{(4)} + \left[P(r_p)^2_{\text{comp}} - (GJ_{\text{k}})_{\text{comp}}\right]\theta'' + Py_0u'' = 0 \end{aligned}\right\} \tag{80}$$

This is the governing equation for the bending and bending of the C-section. The first equation is the twisted flexion balance equation around the weak axis y , while the second equation is the twisted flexion balance equation around the shear center.

6. Boundary Condition

For the fourth order constant coefficient differential equation, there are four undetermined coefficients, requiring four boundary conditions given in advance. Two cases of boundary conditions at each end.

(1) Lateral displacement $u(z)$

Given u, or:

$$\frac{\partial F}{\partial u'} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial F}{\partial u''}\right) = -Pu' - Py_0\theta' - \frac{\mathrm{d}}{\mathrm{d}z}\left[(EI_y)_{\text{comp}}u''\right] = 0 \tag{81}$$

Given u' , or:

$$(EI_y)_{\text{comp}}u'' = 0 \tag{82}$$

(2) Corner $\theta(z)$

Given, θ or:

$$(GJ_{\text{k}})_{\text{comp}}\theta' - P(r_p)^2_{\text{comp}}Py_0u' - \frac{\mathrm{d}}{\mathrm{d}z}(E_1I_{\text{w}}\theta'') = 0 \tag{83}$$

Given θ' , or:

$$(EI_{\text{w}})_{\text{comp}}\theta'' = 0 \tag{84}$$

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Thus, we obtain all the boundary conditions of the C-section.

The above boundary conditions can be used to combine the initially different rod end boundary conditions. The following common combinations are summarized as follows:

Articulated end boundary conditions:

$$u = u'' = \theta = \theta'' = 0 \tag{85}$$

That is, the torsion angle of lateral displacement and around the shear core is zero (geometric boundary condition), and the bending moment around the weak axis and the double moment around the shear core are zero (force boundary condition). y

Fixed-end boundary conditions:

$$u = u' = \theta = \theta' = 0 \tag{86}$$

That is, the torsion angle of lateral displacement and around the shear core is zero (geometric boundary condition), and the rotation angle around the weak axis y and the torsion angle around the shear core are zero (geometric boundary condition).

Free boundary conditions:

$$u'' = \theta'' = 0 \tag{87}$$

That is, the bending moment around the weak axis y and the double moment around the shear core are zero (force boundary condition).

In addition, the following natural boundary conditions should be satisfied:

$$-Pu' - Py_0\theta' - \frac{\mathrm{d}}{\mathrm{d}z}\left[\left(EI_y\right)_{\mathrm{comp}} u''\right] = 0 \tag{88}$$

$$\left(GJ_{\mathrm{k}}\right)_{\mathrm{comp}}\theta' - P\left(r_p\right)^2_{\mathrm{comp}} Py_0 u' - \frac{\mathrm{d}}{\mathrm{d}z}\left(E_1 I_{\mathrm{w}}\theta''\right) = 0 \tag{89}$$

That is, the shear force corresponding to the bending moment around the weak axis and the warping torque corresponding to the double torque around the shear core are zero (force boundary condition). This is the natural boundary condition, the force boundary condition. Obviously, for such complex natural boundary conditions. This is also one of the advantages of studying the buckling problem from the energy variability method.

7. Discussion

Both of the above two equations have a cross-coupling term (the last term), that is, the two equations are coupled and cannot be solved separately. That is, the bending and torsion buckling of the C-shaped shaft are synchronized, and this flexion is torsion flexion.

The coefficients of the cross-coupling term in the two equations are equal, which conforms to the reciprocal law of linear elasticity (Betti's Law), indicating that the above variational derivation is formally correct.

The above equation is a system of homogeneous differential equations of the fourth order constant coefficient, so it is a linear buckling problem.

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