Application of Fractional Fourier Transform in Non-stationary Signal Time Frequency Analysis

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Abstract
In this paper, we discuss the definition and properties of Fractional Fourier Transform (FRFT), and introduce the application of fractional Fourier transform in the field of non-stationary signal processing from the viewpoint of Joint Time-Frequency Analysis. In this paper, the definition, advantages and applications of fractional Fourier transform are analyzed, and the applications of fractional Fourier transform in time-frequency analysis of non-stationary signals are introduced. Then, the future research direction of this field is prospected.

Keywords
Fractional Fourier Transform; Non-stationary Signal; Joint Time-Frequency Analysis.

1. Introduction
Signal is the carrier of information, the basic content of modern signal processing theory, which is the analysis and processing of all kinds of optical or electrical signals. The object of modern signal processing is nonlinear, non-Gauss, non-stationary signal. Non-stationary signals are widely used in communication, radar, automatic control, pattern recognition, underwater acoustic, mechanical vibration, seismic exploration and biomedical applications. The mean value, variance and autocorrelation function of nonstationary signals are functions of time varying with time, the Energy spectrum and power spectrum varying with time. From the point of view of statistics, the difference between non-stationary and stationary signals is that the statistics of a certain order change with time. Most of the actual engineering measurement signals are non-stationary and nonlinear, and the abundant frequency information is contained in the non-stationary signal. The key to the analysis and processing of non-stationary signal is how to express the local, accurate description of its time-frequency distribution characteristics.

Signal processing field, people are most concern is how to signal from access to information, access to information, can be obtained directly from the signal, also can be the signal is transformed to a more useful and reflect the characteristics of the signal domain, in the extraction of the signal information and an important information is the distribution of signal energy. The methods of information analysis include: time domain analysis, frequency domain analysis and time-frequency domain analysis.

(1) The time domain analysis method. The essence of time domain analysis method is to describe the energy distribution of the signal as a function of time t. Using instantaneous power to describe the energy in the unit time of a signal. The disadvantage is that it can only reflect the energy distribution of the signal at the stop time, and can not describe the frequency variation of the signal energy.

(2) The frequency domain analysis. The frequency domain analysis of the signal will describe the energy distribution of the signal as a function of frequency. Using Fourier transform to transform the signal into frequency domain. Signal frequency domain representation of the corresponding energy distribution is
\[|X(\omega)|^2\], It represents the signal energy at the frequency of the signal, i.e., the power spectrum. The disadvantage is that the relationship between signal energy distribution and time can not be described. Therefore, the power spectrum can not define two signal time sequence. This defect cannot cope with nonstationary signal spectrum changes with time.

(3) Joint Time-Frequency Analysis. The time-frequency spectrum can describe how to change with time, is very conducive to the description of spectral components on the dependence of time. Joint time-frequency analysis of instantaneous power and power spectrum of two kinds of information fusion for an energy distribution, in essence, is the signal in time or frequency function is converted to two-dimensional time-frequency joint function.

In summary, time-frequency analysis is a useful tool for non-stationary signal analysis, using time-frequency analysis theory, the non stationary signal energy distribution has been described as a function of time and frequency. Joint Time-Frequency Analysis. The method is based on the Gabor transformation[1-6], short time Fourier transform[7-9] and linear time frequency representation, Wigner-Ville Distribution[10-12], WVD, Choi-Williams distribution (CWD) [13-14], Hough transform [15], wavelet transform [16-17], Fractional Fourier Transform (FRFT) [18-20], etc. The fractional Fourier transform is widely used in optics [18-20], Radar Communication, Biomedical and many other fields. Is currently a hot spot for non-stationary signal time frequency analysis. The following aspects of the definition, physical meaning, advantage and application of fractional Fourier transform are discussed.

2. The definition of Fractional Fourier transform and its physical meaning

2.1 The definition of Fractional Fourier Transform

The mathematical expression of the fractional Fourier transform can be regarded as chirp basis decomposition, in its physical sense, it has the characteristics of time and frequency domain rotation. The order of fractional order transforms from 0 to 1, can show all the characteristics of the signal from the time domain to the frequency domain.

The definition of Fractional Fourier transform is [21]:

\[X_p(\mu) = F_p[x](\mu) = \int_{-\infty}^{\infty} x(t) K_\alpha(t, \mu) dt\]  

(1)

here, \(K_\alpha(t, \mu) = \begin{cases} 
\sqrt{(1 - j \cot \alpha)} e^{j \pi \cot \alpha}, & \alpha \neq n\pi , \\
\delta(t - u), & \alpha = 2n\pi , \\
\delta(t + u), & \alpha = (2n \pm 1)\pi ,
\end{cases}\)

is kernel function of fractional order transformation. Where, \(\alpha = p\pi / 2\), \(p\) is the transform order of FRFT, \(F_p\) is the operator of FRFT.

When \(p = 4n + 1\), \(\alpha \) take \((2n\pi + \pi / 2)\), Fractional Fourier Transform degenerate into Fourier Transform, visibly, Fractional Fourier Transform on a circle of 4.

Discrete Fractional Fourier Transform is defined as:

\[X_p(m) = \tilde{F}_p[x](m) = \sum_{n} \tilde{\vartheta}_p(m,n)x(n)\]  

(2)

\(\tilde{F}_p\) is the Discrete Fractional Fourier Transform operator, \(\tilde{\vartheta}_p(m,n)\) is the kernel matrix for the Fractional Fourier Transform.
2.2 The physical meaning of Fractional Fourier Transform

Figure 1 shows time frequency plane rotation $\alpha$ angle to reach the $(v,u)$ plane, just as the Fourier transformation can be regarded as a signal around the time axis, the axis rotates 90 degrees clockwise until the frequency axis, the fractional Fourier transform can be regarded as a signal of the rotation angle of the time axis to reach the fractional Fourier domain. The characteristics of the fractional Fourier transform can describe the signal in time domain and frequency domain, so it is a unified time-frequency analysis method.

The Fractional Fourier Transformation can be considered as a kind of traditional Fourier transform for energy. The Fractional Fourier transformation is based on the sweep frequency signal chirp in a certain frequency band, such shown in formula (3).

When $u = \sqrt{u/2\pi}$, $t = \sqrt{t/2\pi}$, variable substitution, then (1) becomes:

$$X_{\mu}(\mu) = \mathcal{F}_\alpha \{ x(t) \} = \sqrt{(1 - j \cot \alpha)} \frac{j^\mu \csc \alpha}{2\pi} \int_{-\infty}^{\infty} x(t) e^{j^\mu \csc \alpha \omega t} dt, \alpha \neq n\pi,$$

In order to calculate conveniently, remember

$$A_{\alpha} = \sqrt{(1 - j \cot \alpha)}$$

where, $g(t) = \sqrt{(1 - j \cot \alpha)} e^{j^2 \csc \alpha \omega t}$, $g(t)$ is chirp signal. The variables in Figure 1 are similar to the time frequency variables on the time-frequency plane. $x$ and $\omega$ respectively represent the variables of the $t$ axis and the $\omega$ axis; and $x_u$, $\omega_u$ respectively represent the variables of the $v$ axis, $u$ axis. The relationship exists between variables such as (5):

$$\begin{vmatrix} x_u \\ \omega_u \end{vmatrix} = \begin{bmatrix} \cos \alpha F & \sin \alpha F \\ -\sin \alpha F & \cos \alpha F \end{bmatrix} \begin{vmatrix} x \\ \omega \end{vmatrix}$$

here, $\alpha F = \frac{a\pi}{2}$ is rotation angle. Then, the Fractional Fourier Transform of the signal is equivalent to the signal energy distribution in the time-frequency plane rotation, the fractional Fourier transform is regarded as a kind of rotation operator of the time-frequency plane, which is advantageous to deal with the non-stationary signals. This feature is very favorable for dealing with non-stationary signal analysis and processing.

3. The advantages of processing non-stationary signals by fractional Fourier transform

When the fractional Fourier transform has a clear physical interpretation and practical implementation of the method, the use of FRFT in the field of signal processing applications more and more widely.
Compared to the many advantages of fractional Fourier transform, as a generalized form of the Fourier transform, fractional Fourier transform is in the unity of time and frequency domain signal processing, and the traditional Fourier transform, which is more suitable for processing non-stationary signals and stronger flexibility.

With the angle change of the FRFT, corresponding to the feature information of frequency point can be easily extracted, and aiming at the different time frequency distribution, which can be designed filter [28-29] fractional Fourier transform, which is good to achieve signal noise separation.

(2) As the fractional Fourier transform can be understood as the Chirp based decomposition, therefore, the fractional Fourier transform is especially suitable for the treatment of Chirp type of signal detection, tracking and recognition, has a good effect on Chirp signal.

(3) The fractional Fourier transform has the intrinsic relation with many time frequency analysis methods. Moreover, the fractional Fourier transform is a linear transformation, and there is no cross term interference.

(4) Fractional Fourier transform can also be used as a differential equation, so that the linear differential equation can be transformed into the constant coefficient of the algebraic equation.

(5) Easy optical implementation.

(6) the realization of the fast Fourier transform makes the engineering application of FRFT possible.

Figure 2 and figure 3 is the fractional Fourier transform of a sinusoidal signal and the $sinc(t)$ function ,with the fractional number varied from 0 to 2, to the overall space distribution, from the image can be found that fractional Fourier transform can show signal time-frequency characteristic well.

Fig2. The Fractional Fourier Transformation of the sine signal with the fractional order changes from 0 to 2.

Fig3. Fractional Fourier Transformation of the $sinc(t)$ function with the fractional order changes from 0 to 2.
4. Conclusions

At present, scholars research results on fractional Fourier FRFT mainly: fractional Fourier transform definition, nature; relationship with other time-frequency analysis method; discretization and numerical method for the fractional Fourier transform, fast algorithm; application in the field of bioinformatics, such as heart electric signal, sea clutter, radar, seismic signal detection and feature extraction, image encryption, watermark, face recognition, etc.; high dimensional fractional Fourier transform definition, properties and applications. For the study of the fusion of the fractional Fourier transform and other algorithms, it will broaden and deepen its theory and application field. Fractional transforms is a unified time-frequency transform, and also reflects the signal in time domain and frequency domain information, make up the transform can only reflect global properties of the signal, the spectrum of the whole signal, and can not express the time-frequency localization characteristic of the signal, suitable for processing of the non-stationary signal. In addition to the field of signal processing, the fractional Fourier transform has been applied to quantum mechanics, optical transmission, artificial neural network, control etc..

References