Interval-valued Dual Hesitant Fuzzy Linguistic Arithmetic Aggregation Operators in Multiple Attribute Decision Making

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Abstract

In this paper, we investigate the multiple attribute decision making (MADM) problem based on the arithmetic and geometric aggregation operators with interval-valued dual hesitant fuzzy linguistic information. Then, motivated by the ideal of traditional arithmetic operation, we have developed some aggregation operators for aggregating interval-valued dual hesitant fuzzy linguistic information: interval-valued dual hesitant fuzzy linguistic weighted average (IVDHFLWA) operator, interval-valued dual hesitant fuzzy linguistic ordered weighted average (IVDHFLOWA) operator and Interval-valued dual hesitant fuzzy linguistic hybrid average (IVDHFLHA) operator. Then, we have utilized these operators to develop some approaches to solve the interval-valued dual hesitant fuzzy linguistic multiple attribute decision making problems.

Keywords

Multiple attribute decision making (MADM); interval-valued dual hesitant fuzzy linguistic values; interval-valued dual hesitant fuzzy linguistic hybrid average (IVDHFLHA) operator; interval-valued dual hesitant fuzzy linguistic set.

1. Introduction

Atanassov[1-3]introduced the concept of intuitionistic fuzzy set(IFS), which is a generalization of the concept of fuzzy set[4]. Each element in the IFS is expressed by an ordered pair, and each ordered pair is characterized by a membership degree and a non-membership degree. The sum of the membership degree and the non-membership degree of each ordered pair is less than or equal to 1. The intuitionistic fuzzy set has received more and more attention since its appearance[5-20]. Furthermore, Torra and Narukawa[21] and Torra[22] proposed the hesitant fuzzy set which permits the membership having a set of possible values and discussed the relationship between hesitant fuzzy set and intuitionistic fuzzy set, and showed that the envelope of hesitant fuzzy set is an intuitionistic fuzzy set. Xia and Xu[23] gave an intensive study on hesitant fuzzy information aggregation techniques and their application in decision making. Xu and Xia[24] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. Xu and Xia[25] defined the distance and correlation measures for hesitant fuzzy information and then discuss their properties in detail. Xu et al. [26] developed several series of aggregation operators for hesitant fuzzy information with the aid of quasi-arithmetic means. Gu et al.[27] utilized the hesitant fuzzy weighted averaging (HFWA) operator to investigat the evaluation model for risk investment with hesitant fuzzy information. Motivated by the ideal of prioritized aggregation operators[28], Wei[29] developed some prioritized aggregation operators for aggregating hesitant fuzzy information, and then apply them to develop some models for hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority
level. Wei et al. [30] proposed two hesitant fuzzy Choquet integral aggregation operators: hesitant fuzzy Choquet ordered averaging (HFCOA) operator and hesitant fuzzy Choquet ordered geometric (HFCOG) operator. Wang et al. [31] proposed the generalized hesitant fuzzy hybrid weighted distance (GHFHWDD) measure, which is based on the generalized hesitant fuzzy weighted distance (GHFWD) measure and the generalized hesitant fuzzy ordered weighted distance (GHFOWD) measure [24] and studied some desirable properties of the GHFHWDD measure. Zhu et al. [32] explored the geometric Bonferroni mean (GBM) considering both the BM and the geometric mean (GM) under hesitant fuzzy environment.

From above analysis, we can see that hesitant fuzzy set is a very useful tool to deal with uncertainty. More and more multiple attribute decision making theories and methods under hesitant fuzzy environment have been developed. Current methods are under the assumption that hesitant fuzzy set permits the membership having a set of possible exact and crisp values. However, under many conditions, for the real multiple attribute group decision making problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human think, thus, exact and crisp values are inadequate or insufficient to model real-life decision problems. Indeed, human judgments including preference information may be stated which permits the membership having a set of possible interval-valued dual hesitant fuzzy linguistic values. So, in this paper we shall propose the concept of the interval-valued dual hesitant fuzzy linguistic set based on hesitant fuzzy set to overcome this limitation. In this paper, we investigate the multiple attribute decision making (MADM) problem based on the arithmetic and geometric aggregation operators with interval-valued dual hesitant fuzzy linguistic information. Then, motivated by the ideal of traditional arithmetic operation, we have developed some aggregation operators for aggregating interval-valued dual hesitant fuzzy linguistic information: interval-valued dual hesitant fuzzy linguistic weighted average (IVDHFLWA) operator, interval-valued dual hesitant fuzzy linguistic ordered weighted average (IVDHFLOWA) operator and Interval-valued dual hesitant fuzzy linguistic hybrid average (IVDHFLHA) operator. Then, we have utilized these operators to develop some approaches to solve the interval-valued dual hesitant fuzzy linguistic multiple attribute decision making problems.

2. Hesitant fuzzy linguistic set

2.1 Hesitant fuzzy set

In the following, we briefly describe some basic concepts and basic operational laws related to intuitionistic fuzzy set and hesitant fuzzy sets. Atanassov [1-3] extended the fuzzy set to the intuitionistic fuzzy set (IFS), shown as follows.

Definition 1. An IFS \( A \) in \( X \) is given by

\[
A = \{ (x, \mu_A (x), \nu_A (x)) | x \in X \}
\]  

(1)

Where \( \mu_A : X \rightarrow [0,1] \) and \( \nu_A : X \rightarrow [0,1] \), with the condition \( 0 \leq \mu_A (x) + \nu_A (x) \leq 1, \forall x \in X \). The numbers \( \mu_A (x) \) and \( \nu_A (x) \) represent, respectively, the membership degree and non-membership degree of the element \( x \) to the set \( A \) [1-3].

However, when giving the membership degree of an element, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. For such cases, Torra [22] proposed another generation of FS.

Definition 2[22]. Given a fixed set \( X \), then a hesitant fuzzy set (HFS) on \( X \) is in terms of a function that when applied to \( X \) returns a sunset of \([0,1]\), the HFS can be expressed by mathematical symbol:

\[
E = \{ (x, h_k (x)) | x \in X \}
\]  

(2)
where $h_E(x)$ is a set of some values in $[0,1]$, denoting the possible membership degree of the element $x \in X$ to the set $E$.

For convenience, Xia and Xu[23] called $h = h_E(x)$ a hesitant fuzzy element (HFE) and $H$ the set of all HFEs.

**Definition 3[23].** For a HFE $h$, $s(h) = \frac{1}{\# h} \sum_{\gamma \in h} \gamma$ is called the score function of $h$, where $\# h$ is the number of the elements in $h$. For two HFEs $h_1$ and $h_2$, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Based on the relationship between the HFEs and IFVs, Xia and Xu[23] define some new operations on the HFEs $h$, $h_1$ and $h_2$:

1. $h^i = \bigcup_{\gamma \in h} \{\gamma^i\}$;
2. $\lambda h = \bigcup_{\gamma \in h} \{1-(1-\gamma)^i\}$; 
3. $h_1 \oplus h_2 = \bigcup_{\gamma_1, \gamma_2 \in h_1, h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$;
4. $h_1 \otimes h_2 = \bigcup_{\gamma_1, \gamma_2 \in h_1, h_2} \{\gamma_1 \gamma_2\}$.

### 2.2 Linguistic term set

Let $S = \{s_i | i = 1, 2, \ldots, t\}$ be a linguistic term set with odd cardinality. Any label, $s_i$ represents a possible value for a linguistic variable, and it should satisfy the following characteristics[33-44]:

1. The set is ordered: $s_i > s_j$, if $i > j$;
2. There is the negation operator: $\text{neg}(s_i) = s_j$ such that $i + j = t + 1$;
3. Max operator: $\max(s_i, s_j) = s_i$, if $s_i > s_j$;
4. Min operator: $\min(s_i, s_j) = s_j$, if $s_i < s_j$.

For example, $S$ can be defined as

$S = \{s_i = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{medium}, s_5 = \text{good}, s_6 = \text{very good}, s_7 = \text{extremely good}\}$

To preserve all the given information, we extend the discrete term set $S$ to a continuous term set $	ilde{S} = \{s_a | s_a \leq s \leq s_\beta, a \in [1,q]\}$, where $q$ is a sufficiently large positive integer. If $s_a \in S$, then we call $s_a$ the original linguistic term, otherwise, we call $s_a$ the virtual linguistic term. In general, the decision maker uses the original linguistic term to evaluate attributes and alternatives, and the virtual linguistic terms can only appear in calculation [36-37].

Consider any two linguistic variables $s_a$ and $s_\beta$, $s_a, s_\beta \in \tilde{S}, \lambda \in [0,1]$ we define their operational laws as follows [36-37]:

1. $s_a \oplus s_\beta = s_{a+\beta}$;
2. $s_a \ominus s_\beta = s_{a-\beta}$;
3. $\lambda s_a = s_{a\lambda}$;
4. $(s_a)^2 = s_a$. 


2.3 Hesitant fuzzy linguistic set

**Definition 4.** Given a fixed set $X$, then a hesitant fuzzy linguistic set (HFLS) on $X$ is in terms of a function that when applied to $X$ returns a sunset of $[0,1]$. To be easily understood, the HFLS can be expressed by mathematical symbol as follows:

$$A = \left\{ \left( x, s_{\theta(x)}, h_A(x) \right) \mid x \in X \right\}$$  \hspace{1cm} (3)

where $h_A(x)$ is a set of some values in $[0,1]$, denoting the possible membership degree of the element $x \in X$ to the linguistic set $s_{\theta(x)}$. For convenience, we called $a = \left( s_{\theta(x)}, h_A(x) \right)$ a hesitant fuzzy linguistic element (HFLE) and $A$ the set of all HFLEs.

**Definition 5.** For a HFLE $a = \left( s_{\theta(x)}, h_A(x) \right)$, $s(a) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma s_{\theta(x)}$ is called the score function of $a$, where $\#h$ is the number of the elements in $h$. For two HFLEs $a_1$ and $a_2$, if $s(a_1) > s(a_2)$, then $a_1 > a_2$; if $s(a_1) = s(a_2)$, then $a_1 = a_2$.

Based on the relationship between the HFLEs, we shall define some new operations on the HFLEs $a = \left( s_{\theta(a)}, h(a) \right)$, $a_1 = \left( s_{\theta(a_1)}, h(a_1) \right)$ and $a_2 = \left( s_{\theta(a_2)}, h(a_2) \right)$:

1. $a^i = \left\{ s_{\theta(a^i)}, \bigcup_{\gamma(a) \in h(a)} \left\{ \gamma\left( a^i \right) \right\} \right\}$;
2. $\lambda a = \left\{ s_{\lambda \theta(a)}, \bigcup_{\gamma(a) \in h(a)} \left\{ 1 - (1 - \gamma(a)) \right\} \right\}$;
3. $a_1 \oplus a_2 = \left\{ s_{\theta(a_1) \oplus \theta(a_2)}, \bigcup_{\gamma(a) \in h(a_1), \gamma(a_2) \in h(a_2)} \left\{ \gamma(a_1) + \gamma(a_2) - \gamma(a_1) \gamma(a_2) \right\} \right\}$;
4. $a_1 \otimes a_2 = \left\{ s_{\theta(a_1) \otimes \theta(a_2)}, \bigcup_{\gamma(a) \in h(a_1), \gamma(a_2) \in h(a_2)} \left\{ \gamma(a_1) \gamma(a_2) \right\} \right\}$.

3. **Interval-valued Dual Hesitant Fuzzy Linguistic Arithmetic Aggregation operators**

From above analysis, we can see that hesitant fuzzy set is a very useful tool to deal with uncertainty. More and more multiple attribute decision making theories and methods under hesitant fuzzy environment have been developed. Current methods are under the assumption that hesitant fuzzy set permits the membership having a set of possible exact and crisp values. However, under many conditions, for the real multiple attribute group decision making problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human think, thus, exact and crisp values are inadequate or insufficient to model real-life decision problems. Indeed, human judgments including preference information may be stated which permits the membership having a set of possible interval-valued dual hesitant fuzzy linguistic values. In the following, we shall propose some basic concepts and basic operational laws related to interval-valued dual hesitant fuzzy linguistic set.

**Definition 6.** Given a fixed set $X$, then an interval-valued dual hesitant fuzzy linguistic set (IVDHFLS) on $X$ is in terms of a function that when applied to $X$ returns a sunset of $[0,1]$. To be easily understood, the IVDHFLS can be expressed by mathematical symbol as follows:

$$A = \left\{ \left( x, s_{\theta(x)}, \tilde{h}_A(x), \tilde{g}(x) \right) \mid x \in X \right\}$$  \hspace{1cm} (4)
which \( \tilde{h}(x) \) and \( \tilde{g}(x) \) are two sets of some interval values in \([0,1]\), denoting the possible membership degree and non-membership degrees of the element \( x \in X \) to the linguistic set \( s_{(\tilde{h}(x))} \) respectively.

For convenience, we called \( a = \left\{ s_{(\tilde{h}(x))} (\tilde{h}_A(x), \tilde{g}(x)) \right\} \) an interval-valued dual hesitant fuzzy linguistic element (IVDHFLE) and \( A \) the set of all IVDHFLEs.

**Definition 7.** Let \( \tilde{a} = \left\{ a^L, a^R \right\} = \{ x | a^L \leq x \leq a^R \} \), then \( \tilde{a} \) is called an interval fuzzy number. If \( 0 \leq a^L \leq a^R \), then \( \tilde{a} \) is called a positive interval fuzzy number.

**Definition 8.** If \( \tilde{h} = \left[ y^L, y^R \right] (y^R \geq y^L \geq 0) \) is interval number, then the expected value of \( \tilde{h} \) is

\[
E(\tilde{h}) = \frac{1}{2} (y^L + y^R)
\]

**Definition 9.** For three IVDHFLE \( \tilde{a} = \left\{ s_{(\tilde{h}(x))} (\tilde{h}_A(x), \tilde{g}(x)) \right\} \), \( \tilde{a}_1 = \left\{ s_{(\tilde{h}(x))} (\tilde{h}_A(x), \tilde{g}(x)) \right\} \) and \( \tilde{a}_2 = \left\{ s_{(\tilde{h}(x))} (\tilde{h}_A(x), \tilde{g}(x)) \right\} \), \( s(\tilde{a}) = \frac{1 + E\left( \frac{1}{\# h} \sum_{j=1}^{\# h} \tilde{g} \right) - E\left( \frac{1}{\# g} \sum_{j=1}^{\# g} \tilde{h} \right) }{2} s_{(\tilde{h}(x))} \) is called the score function of \( \tilde{a} \), and \( p(\tilde{a}) = \left( E\left( \frac{1}{\# h} \sum_{j=1}^{\# h} \tilde{g} \right) + E\left( \frac{1}{\# g} \sum_{j=1}^{\# g} \tilde{h} \right) \right) s_{(\tilde{h}(x))} \) the accuracy function of \( \tilde{a} \).

where \( \# h \) and \( \# g \) are the numbers of the elements in \( \tilde{h} \) and \( \tilde{g} \) respectively, then

- If \( s(\tilde{a}_1) > s(\tilde{a}_2) \), then \( \tilde{a}_1 \) is superior to \( \tilde{a}_2 \), denoted by \( \tilde{a}_1 \succ \tilde{a}_2 \);
- If \( s(\tilde{a}_1) = s(\tilde{a}_2) \), then \( \tilde{a}_1 \) is equivalent to \( \tilde{a}_2 \), denoted by \( \tilde{a}_1 \equiv \tilde{a}_2 \);
- If \( p(\tilde{a}_1) > p(\tilde{a}_2) \), then \( \tilde{a}_1 \) is superior to \( \tilde{a}_2 \), denoted by \( \tilde{a}_1 \succ \tilde{a}_2 \).

Motivated by the operational law of interval-valued dual hesitant fuzzy linguistic set, in the following, we shall develop some interval-valued dual hesitant fuzzy linguistic arithmetic aggregation operator based on the operations of IVDHFLEs.

**Definition 10.** Let \( \tilde{a}_j = \left\{ s_{(\tilde{h}_j(x))} (\tilde{h}_j(x), \tilde{g}(x)) \right\} (j = 1, 2, \ldots, n) \) be a collection of IVDHFLEs, then we define the interval-valued dual hesitant fuzzy linguistic weighted average (IVDHFLWA) operator as follows:

\[
\text{IVDHFLWA}_\omega (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \bigoplus_{j=1}^{n} (\omega_j \tilde{a}_j)
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the weight vector of \( \omega_j (j = 1, 2, \ldots, n) \), and \( \omega_j > 0 \), \( \sum_{j=1}^{n} \omega_j = 1 \).

Based on operations of the interval-valued dual hesitant fuzzy linguistic values described, we can drive the Theorem 1.

**Theorem 1.** Let \( \tilde{a}_j = \left\{ s_{(\tilde{h}_j(x))} (\tilde{h}_j(x), \tilde{g}(x)) \right\} (j = 1, 2, \ldots, n) \) be a collection of IVDHFLEs, then their aggregated value by using the IVDHFLWA operator is also an IVDHFLE, and
IVDHFLWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \\
= \bigoplus_{j=1}^{n} \left( \omega_j \tilde{a}_j \right) \\
= \sum_{j=1}^{n} \omega_j s_{\delta_{\hat{a}_j}} \left( \bigcup_{j, j_1, j_2, \cdots, j_n} \left[ 1 - \prod_{j=1}^{n} \left( 1 - \mu_{j_1} \right)^{\omega_{j_1}}, 1 - \prod_{j=1}^{n} \left( 1 - \mu_{j_2} \right)^{\omega_{j_2}} \right], \prod_{j=1}^{n} (v_{j_1})^{w_{j_1}}, \prod_{j=1}^{n} (v_{j_2})^{w_{j_2}} \right) \right) \right) \right) \right) \\
\]where \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \) be the weight vector of \( \tilde{a}_j \) \( (j = 1, 2, \cdots, n) \), and \( \omega_j > 0, \sum_{j=1}^{n} \omega_j = 1 \) and \( \check{h}_j = [\mu_{j_1}, \mu_{j_2}], \check{g}_j = [v_{j_1}, v_{j_2}] \).

**Definition 11.** Let \( \tilde{a}_j = \left( s_{\delta_{\hat{a}_j}} \right) (\check{h}_j, \check{g}_j) \) \( (j = 1, 2, \cdots, n) \) be a collection of IVDHFLEs, then we define the interval-valued dual hesitant fuzzy linguistic ordered weighted average (IVDHFLOWA) operator as follows:

\[ \text{IVDHFLOWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \bigoplus_{j=1}^{n} \left( w_j \tilde{a}_{\sigma(j)} \right) \]

where \( (\sigma(1), \sigma(2), \cdots, \sigma(n)) \) is a permutation of \( (1, 2, \cdots, n) \), such that \( \tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)} \) for all \( j = 2, \cdots, n \), and \( w = (w_1, w_2, \cdots, w_n)^T \) is the aggregation-associated weight vector such that \( w_j \in [0,1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

Based on operations of the interval-valued dual hesitant fuzzy linguistic values described, we can drive the Theorem 2.

**Theorem 2.** Let \( \tilde{a}_j = \left( s_{\delta_{\hat{a}_j}} \right) (\check{h}_j, \check{g}_j) \) \( (j = 1, 2, \cdots, n) \) be a collection of IVDHFLEs, then their aggregated value by using the IVDHFLOWA operator is also an IVDHFLE, and

\[ \text{IVDHFLOWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \bigoplus_{j=1}^{n} \left( w_j \tilde{a}_{\sigma(j)} \right) \]

\[ = \sum_{j=1}^{n} w_j s_{\delta_{\hat{a}_{\sigma(j)}}} \left( \bigcup_{j, j_1, j_2, \cdots, j_n} \left[ 1 - \prod_{j=1}^{n} \left( 1 - \mu_{j_1} \right)^{w_{j_1}}, 1 - \prod_{j=1}^{n} \left( 1 - \mu_{j_2} \right)^{w_{j_2}} \right], \prod_{j=1}^{n} (v_{j_1})^{w_{j_1}}, \prod_{j=1}^{n} (v_{j_2})^{w_{j_2}} \right) \right) \right) \right) \right) \right) \right) \]

where \( (\sigma(1), \sigma(2), \cdots, \sigma(n)) \) is a permutation of \( (1, 2, \cdots, n) \), such that \( \tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)} \) for all \( j = 2, \cdots, n \), and \( w = (w_1, w_2, \cdots, w_n)^T \) is the aggregation-associated weight vector such that \( w_j \in [0,1] \) and \( \sum_{j=1}^{n} w_j = 1 \) and \( \check{h}_j = [\mu_{j_1}, \mu_{j_2}], \check{g}_j = [v_{j_1}, v_{j_2}] \).

From Definitions 10 and 11, we know that the IVDFLHWA operator weights the interval-valued dual hesitant fuzzy linguistic argument itself, while the IVDFLOWA operator weights the ordered positions of the interval-valued dual hesitant fuzzy linguistic arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the IVDHFLWA and IVDHFLOWA operators. However, both the operators consider only one of them. To solve this
drawback, in the following we shall propose an interval-valued dual hesitant fuzzy linguistic hybrid average (IVDHFLHA) operator.

**Definition 12.** An interval-valued dual hesitant fuzzy linguistic hybrid average (IVDHFLHA) operator is defined as follows:

$$
IVDHFLHA_{w,\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \oplus_{j=1}^{n} \left( w_j \hat{\alpha}_{\sigma(j)} \right)
$$

(10)

where $w=(w_1, w_2, \ldots, w_n)$ is the associated weighting vector, with $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, and $\hat{\alpha}_{\sigma(j)}$ is the $j$-th largest element of the interval-valued dual hesitant fuzzy linguistic arguments $\hat{a}_{\sigma(j)} \left( \hat{a}_{\sigma(j)} = n\omega_j \tilde{a}_j, j = 1, 2, \ldots, n \right)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of interval-valued dual hesitant fuzzy linguistic arguments $\tilde{a}_i \left( i = 1, 2, \ldots, n \right)$, with $\omega_i \in [0,1]$, $\sum_{i=1}^{n} \omega_i = 1$, and $n$ is the balancing coefficient. Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then IVDHFLHA is reduced to the interval-valued dual hesitant fuzzy linguistic weighted average (IVDHFLWA) operator; if $\omega = (1/n, 1/n, \ldots, 1/n)$, then IVDHFLHA is reduced to the interval-valued dual hesitant fuzzy linguistic ordered weighted average (IVDHFLOWA) operator.

Based on operations of the interval-valued dual hesitant fuzzy linguistic values described, we can drive the Theorem 3.

**Theorem 3.** Let $\tilde{a}_j = \left( s_{\tilde{a}_j}, (\tilde{h}_j, \tilde{g}_j) \right) (j = 1, 2, \ldots, n)$ be a collection of IVDHFLEs, then their aggregated value by using the IVDHFLHA operator is also an IVDHFLE, and

$$
IVDHFLHA_{w,\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)
$$

(11)

$$
= \oplus_{j=1}^{n} \left( w_j \hat{\alpha}_{\sigma(j)} \right)
$$

$$
= \left\{ \begin{array}{ll} 
\sum_{j=1}^{n} w_j \sum_{i=1}^{n} \left[ 1 - \prod_{j=1}^{n} \left( 1 - \mu_{\sigma(j)}^{i} \right)^{w_j} \right] \right. \\
\left. \prod_{j=1}^{n} \left( \nu_{\sigma(j)}^{i} \right)^{w_j} \prod_{j=1}^{n} \left( \nu_{\sigma(j)}^{i} \right)^{w_j} \right] \end{array} \right.
$$

where $w=(w_1, w_2, \ldots, w_n)$ is the associated weighting vector, with $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, and $\hat{\alpha}_{\sigma(j)}$ is the $j$-th largest element of the interval-valued dual hesitant fuzzy linguistic arguments $\hat{a}_{\sigma(j)} \left( \hat{a}_{\sigma(j)} = n\omega_j \tilde{a}_j, j = 1, 2, \ldots, n \right)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of interval-valued dual hesitant fuzzy linguistic arguments $\tilde{a}_i \left( i = 1, 2, \ldots, n \right)$, with $\omega_i \in [0,1]$, $\sum_{i=1}^{n} \omega_i = 1$, and $n$ is the balancing coefficient.

4. **An approach to multiple attribute decision making with interval-valued dual hesitant fuzzy linguistic information**

In this section, we shall utilize the interval-valued dual hesitant linguistic aggregation operators to multiple attribute decision making with interval-valued dual hesitant fuzzy linguistic information.
The following assumptions or notations are used to represent the MADM problems with interval-valued dual hesitant fuzzy linguistic information. Let \( A = \{ A_1, A_2, \ldots, A_n \} \) be a discrete set of alternatives, and \( G = \{ G_1, G_2, \ldots, G_n \} \) be the state of nature. If the decision makers provide several values for the alternative \( A_i \) under the attribute \( G_j \) with respect to \( s_{ij} \) with anonymity, these values can be considered as an interval-valued dual hesitant fuzzy linguistic element \( \left( s_{ij}, \left( \bar{h}_{ij}, \bar{g}_{ij} \right) \right) \). In the case where two decision makers provide the same value, then the value emerges only once in \( \left( \bar{h}_{ij}, \bar{g}_{ij} \right) \).

Suppose that the decision matrix \( H = \left( \bar{h}_{ij} \right)_{m \times n} = \left( \left( s_{ij}, \left( \bar{h}_{ij}, \bar{g}_{ij} \right) \right) \right) \) is the interval-valued dual hesitant fuzzy linguistic decision matrix, where \( \left( s_{ij}, \left( \bar{h}_{ij}, \bar{g}_{ij} \right) \right) (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n) \) are in the form of DHFLEs.

In the following, we apply the IVDHFLWA operator to the MADM problems with interval-valued dual hesitant fuzzy linguistic information.

**Step 1.** We utilize the decision information given in matrix \( H \), and the IVDHFLWA operator

\[
\bar{h}_i = \left( \left( s_{ij}, \left( \bar{h}_{ij}, \bar{g}_{ij} \right) \right) \right) = \text{IVDHFLWA}_i \left( \bar{h}_{i1}, \bar{h}_{i2}, \ldots, \bar{h}_{in} \right), i = 1, 2, \ldots, m.
\]

(12)

to derive the overall preference values \( \bar{h}_i (i = 1, 2, \ldots, m) \) of the alternative \( A_i \).

**Step 2.** Calculate the scores \( S \left( \bar{h}_i \right) (i = 1, 2, \ldots, m) \) of the overall interval-valued dual hesitant fuzzy linguistic preference values \( h_i (i = 1, 2, \ldots, m) \) to rank all the alternatives \( A_i (i = 1, 2, \ldots, m) \) and then to select the best one(s). If there is no difference between two scores \( s \left( \bar{h}_i \right) \) and \( s \left( \bar{h}_j \right) \), then we need to calculate the accuracy degrees \( p \left( \bar{h}_i \right) \) and \( p \left( \bar{h}_j \right) \) of the overall interval-valued dual hesitant fuzzy linguistic numbers \( \bar{h}_i \) and \( \bar{h}_j \), respectively, and then rank the alternatives \( A_i \) and \( A_j \) in accordance with the accuracy degrees \( p \left( \bar{h}_i \right) \) and \( p \left( \bar{h}_j \right) \).

**Step 3.** Rank all the alternatives \( A_i (i = 1, 2, \ldots, m) \) and select the best one(s) in accordance with \( S \left( \bar{h}_i \right) (i = 1, 2, \ldots, m) \).

**Step 4.** End.

5. Conclusion

In this paper, we investigate the multiple attribute decision making (MADM) problem based on the arithmetic and geometric aggregation operators with interval-valued dual hesitant fuzzy linguistic information. Then, motivated by the ideal of traditional arithmetic operation[45-48], we have developed some aggregation operators for aggregating interval-valued dual hesitant fuzzy linguistic information: interval-valued dual hesitant fuzzy linguistic weighted average (IVDHFLWA) operator, interval-valued dual hesitant fuzzy linguistic ordered weighted average (IVDHFLOWA) operator and interval-valued dual hesitant fuzzy linguistic hybrid average (IVDHFLHA) operator. Then, we have utilized these operators to develop some approaches to solve the interval-valued dual hesitant fuzzy linguistic multiple attribute decision making problems.
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